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FAMILIES OF DIRAC OPERATORS, BOUNDARIES AND THE B-CALCULUS

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Abstract

A version of the Atiyah-Patodi-Singer index theorem is proved for general families of Dirac operators on compact manifolds with boundary. The vanishing of the analytic index of the boundary family, in K^1 of the base, allows us to define, through an explicit trivialization, a smooth family of boundary conditions of generalized Atiyah-Patodi-Singer type. The calculus of b-pseudodifferential operators is then employed to establish the family index formula. A relative index formula, describing the effect of changing the choice of the trivialization, is also given. In case the boundary family is invertible the form of the index theorem obtained by Bismut and Cheeger is recovered.

Introduction

Let $\phi : M \longrightarrow B$ be a smooth fibration of a manifold with boundary, M, with compact fibres diffeomorphic to a fixed manifold with boundary, X. In case the fibres are even-dimensional, carry smoothly varying spin structures and metrics which are of product type near the boundary and the Dirac operators induced on the fibres of the boundary fibration are all invertible, Bismut and Cheeger [9] obtained a family version of the Atiyah-Patodi-Singer index theorem:

$$\mathrm{Ch}(\mathrm{Ind}) = \phi_*(\widehat{A}) - \frac{1}{2}\widehat{\eta}. \tag{1}$$

Here Ch(Ind) is the Chern character, in $H^{\mathrm{ev}}(B)$, of the virtual bundle formed by the $\mathbb{Z}_2$-graded null spaces of the family of Dirac operators,

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$\eth$, with Atiyah-Patodi-Singer boundary condition, $\phi_*(\widehat{A})$ is the push-forward (i.e., fibre integral) of the $\widehat{A}$-genus of M/B, and $\widehat{\eta}$ is a form on B determined by the family of Dirac operators on the boundary. In [11] a similar formula is presented, without proof, when the boundary family has a null space of constant dimension. Although this applies to the interesting case of the signature operator, it is inadequate for many potential applications, for example to gauge theory. As an example consider the family of Dirac operator parametrized by the space of unitary connections on a Hermitian vector bundle over a punctured Riemann surface; this case has been studied in the context of determinant bundles by Chang [14] where generalized Atiyah-Patodi-Singer boundary conditions were introduced.

In this paper a general family index theorem of Atiyah-Patodi-Singer type is derived with no assumptions placed on the family of Dirac operators induced on the boundaries of the fibres. Namely we show that

$$\mathrm{Ch}(\mathrm{Ind}_P) = \phi_*(\mathrm{AS}) - \frac{1}{2}\widehat{\eta}_P \quad \text{in } H^*(B). \tag{2}$$

Here the virtual index bundle, Ind_P, arises from a smooth family of boundary conditions of generalized Atiyah-Patodi-Singer type, represented by the spectral section P. By this we mean that P is a smooth family of projections acting on the spaces of square-integrable sections over each of the fibres of the boundary fibration with each projection annihilating the eigenspaces of the corresponding boundary Dirac operators of sufficiently negative eigenvalues and acting as the identity on eigenspaces of sufficiently positive eigenvalues. Let D be an arbitrary family of self-adjoint elliptic operators associated to a fibration $\psi : M' \longrightarrow B$ with compact fibres diffeomorphic to the manifold Y, $\partial Y = \emptyset$. The existence of a spectral section for D is shown below to be equivalent to the vanishing of the K^1-index of the family in the sense of Atiyah and Singer [7]. A spectral section can exist without the family D having a spectral gap; if one spectral section exists then there are infinitely many. If the family D arises as the boundary of a family of Dirac operators, as in the case of main interest here, then the K^1-index is always zero and therefore a spectral section P exists. In case the boundary family is invertible, P can be taken as the projection onto the positive part of the spectrum and this gives the Atiyah-Patodi-Singer boundary condition on each fibre; then (2) reduces to (1). The eta form on the right in (2) is defined by global integrals on the boundary fibres, see (14.5), essentially as given by Bismut and Cheeger in [9], of a perturbation of the boundary superconnection induced by Bismut's form of the Levi-Civita superconnection. The spectral section, P, is involved since it determines which perturbations are admissible; as a result $\widehat{\eta}_P$ is determined only up to an exact form. There is a second, more minor, sense in which (2) is more general than (1) in that we treat the Dirac operators corresponding to $\mathbb{Z}_2$-graded Hermitian Clifford modules over the fibres, with graded unitary connections. The Atiyah-Singer class AS is then a product of $\widehat{A}$ with a characteristic class arising from the non-Clifford part of the bundle.

Bismut in [8] carried out the program, initiated by Quillen [22], of deriving the family index theorem for Dirac operators using the notion of a superconnection. Here we construct the Bismut superconnection in the setting of b-geometry relying heavily on the treatment of the boundaryless case by Berline, Getzler and Vergne in [13, Chapter 10] including the analysis of the long-time behaviour of the supertrace given by Berline and Vergne [12]. We also make considerable use of the discussion in [20] of the single-operator case. Our proof of (2) uses heat equation methods, as do most proofs of related results starting with [4]. As in [20] we take the position (as noted there, already implicit in [4]) that Atiyah-Patodi-Singer index theorems are best viewed, and directly proved, in terms of the index theory of operators on complete Riemannian manifolds with (asymptotically) cylindrical ends. This is to be contrasted with the explicit approach in [4] that these are index theorems on incomplete manifolds, with metrics which are of product type near the boundary, with only the proofs using extension across the boundary to a complete metric with cylindrical end. On the other hand Bismut and Cheeger, in [9], use a somewhat different method of proof in which the incomplete manifold is extended to a singular (still incomplete) space and then the formula is recovered by an 'adiabatic limit' to the complete case. Although these various points of view are largely interchangable our carrying out the proof in the context of the exact b-metrics, discussed in [20], is not only because these are slightly more general than product cylindrical ends. More significantly we use the analysis which arises naturally in the context of b-metrics, as recalled in the Appendix, both to perturb the boundary operator to make the family Fredholm and also to regularize the virtual index bundle of null spaces to a true bundle on these complete manifolds. The eta forms then arise as the trace anomaly for the extensions of the trace to the algebra of b-pseudodifferential operators and to exploit this we need to show that the heat kernel of the perturbed operator is of this type.

In Section 1 the geometry of fibrations with fibres which are even-dimensional manifolds with boundary is described in the context of b-geometry, in the sense of [20]. In Section 2 the existence of a spectral section for a family of first-order self-adjoint elliptic differential operators is shown to be equivalent to the vanishing of the index as a K^1 class on the base; the cobordism invariance of this class is then proved showing the existence of one spectral section (and therefore infinitely many) of our boundary family.

In the conventional setting of a family of metrics with product decomposition near the boundary the notion a generalized Atiyah-Patodi-Singer boundary condition, determined by a spectral section of the boundary family, is described in Section 3. The related boundary conditions for Dirac operators associated to metrics with cylindrical ends are introduced in Section 4 and in Section 5 for the general case of a family of exact b-metrics. This leads to the definition of the index bundle and its stabilization in Section 6. The regularization of the boundary problem, by the addition of smoothing terms, is discussed in Section 8. For two spectral sections for the same family the relative index theorem in Section 7 expresses the difference of the index classes in terms of the Chern character of a virtual bundle over the boundary. The Bismut superconnection, in this b-setting, is described in Section 9. For the odd-dimensional case Cl(1)-superconnections in the sense of Quillen, [22], are introduced in Section 10 and related to the boundary behaviour of the Bismut superconnection. The properties of heat kernels related to the superconnection are discussed in Section 11, using results from [20]. In particular the b-trace and its supertrace analogue are described and in Section 12 the defect formula for the b-supertrace is derived. This is used in Section 13 to analyze the Chern character of the rescaled Bismut superconnection, the variation formula for which involves the $\widehat{\eta}$ form. Normalization of the eta form with respect to a spectral section is described in Section 14 and the family index formula, for Dirac operators on even-dimensional fibres, is proved in Section 15. The dependence of the eta forms on the spectral section is discussed in general in Section 16 and used to relate the formula (2) to the special case of a boundary family with constant rank null space from [11]. In the appendix a more technical discussion of the b-pseudodifferential calculus, and the corresponding heat calculus, is presented.

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1. Geometric preliminaries

Let $\phi : M \longrightarrow B$ be a smooth fibration of a manifold with boundary, M, with compact fibres diffeomorphic to the even-dimensional compact manifold with boundary X. Let ${}^bTM$ be the b-tangent bundle over the total space; see [20]. The differential of ϕ extends by continuity from the interior to the b-differential ${}^b\phi_* : {}^bTM \longrightarrow TB$. This also follows from the fact that ϕ is a b-map; see [18]. The null space of ${}^b\phi_*$ is a subbundle

$$
{}^bT(M/B) \subset {}^bTM, \tag{1.1}
$$

which restricts to each fibre $M_z = \phi^{-1}(z)$ to be canonically isomorphic to ${}^bTM_z$. By a family of exact b-metrics on M, $g_{M/B}$, we mean a metric on ${}^bT(M/B)$ which restricts to each fibre to an exact b-metric. In particular there is a defining function $\rho \in \mathcal{C}^\infty(M)$ for the boundary of M such that

$$
g_{M/B} = \frac{d\rho^2}{\rho^2} + g'_{M/B}, \tag{1.2}
$$

where $g'_{M/B}$ is a smooth 2-cotensor on $T(M/B)$, the null space of the differential ϕ_* in the usual sense. The defining function ρ is fixed by (1.2) up to a transformation

$$
\tilde{\rho} = (\phi^* a)\rho + O(\rho^2),\ 0 < a \in \mathcal{C}^\infty(B) \tag{1.3}
$$

near each component of ∂M; see [20, §2.3].

Let ${}^bT^*(M/B)$ be the dual bundle to ${}^bT(M/B)$. The Clifford algebras of these Euclidean bundles forms a bundle over M, $\mathrm{Cl}_\phi(M)$, which is $\mathbb{Z}_2$-graded:

$$
\mathrm{Cl}_\phi(M) = \mathrm{Cl}^+_\phi(M) \oplus \mathrm{Cl}^-_\phi(M), \tag{1.4}
$$

and the induced Levi-Civita connection preserves the grading (see [20, §3.4]. Our convention for the Clifford algebra is that $\xi\eta + \eta\xi = 2\langle \xi, \eta \rangle$. A superbundle $E \longrightarrow M$,

$$
E = E^+ \oplus E^- \tag{1.5}
$$

is a family of graded Clifford modules if it has a non-trivial smooth fibre action of $\mathrm{Cl}_\phi(M)$. If the bundle is Hermitian and the action is unitary, E is said to be an Hermitian (fibre) Clifford module.

When E has a fibre b-connection, ${}^b\nabla^E$, which distributes over the Clifford action, with respect to the Levi-Civita connection, the bundle has a family of Dirac operators acting on the fibres. Our convention for the associated Dirac operator is slightly different from that of [13], rather follows [20], in that with respect to an orthonormal basis e_k of ${}^bT(M/B)$, with dual basis e^k of ${}^bT^*(M/B)$,

$$\eth u = -i \sum_k \mathrm{cl}(e^k) \cdot {}^b\nabla^E_{e_k} u. \tag{1.6}$$

As in [20, §3.10] it follows that $\eth$ is a family of elliptic b-differential operators on the fibres; in terms of the notation explained in the Appendix

$$\eth \in \mathrm{Diff}^1_{b,\phi}(M; E). \tag{1.7}$$

In fact $\eth$ is formally self-adjoint and odd with respect to the $\mathbb{Z}_2$ grading:

$$\eth = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix}. \tag{1.8}$$

In this paper, unless otherwise stated, we will assume that

$${}^b\nabla^E_{\rho\frac{\partial}{\partial\rho}} \equiv 0 \quad \text{at } \partial M. \tag{1.9}$$

This property is shown in [20, §2.13] to be equivalent to requiring that ${}^b\nabla^E$ be induced by a true connection. Thus (1.9) implies that the curvature tensor is an ordinary smooth (vertical) form with values in the bundle of endomorphisms of E; this, in turn, implies the smoothness of any characteristic class associated to ${}^b\nabla^E$ via Chern-Weil theory. The Levi-Civita connection associated to an exact b-metric satisfies (1.9).

The fibration, ϕ, restricts to a fibration $\partial\phi : \partial M \longrightarrow B$, with fibres diffeomorphic to ∂X. The exact b-metric induces a metric $g_{\partial M/B}$ on the fibres of $\partial\phi$. Let $\mathrm{Cl}_{\partial\phi}(\partial M)$ be the Clifford bundle for the metric on $T^*(\partial M/B)$. The orthogonal embedding $T^*(\partial M/B) \hookrightarrow {}^bT^*_{\partial M}(M/B)$ gives rise to a map

$$T^*(\partial M/B) \ni \xi \longmapsto i\frac{d\rho}{\rho} \cdot \xi \in \mathrm{Cl}_\phi(M), \tag{1.10}$$

which extends to an isomorphism of algebras

$$\mathrm{Cl}_{\partial\phi}(\partial M) \longrightarrow \mathrm{Cl}^+_{\phi,\partial M}(M). \tag{1.11}$$

In particular (1.10) gives an action of $\mathrm{Cl}_{\partial\phi}(\partial M)$ on $E^0 = E^+_{\partial M}$. As in [20] we denote this identification (here by definition) $M^+ : E^+_{\partial M} \longrightarrow E^0$. Then we identify the negative part of the superbundle with E^0 by

$$M^- = M^+ \circ \mathrm{cl}(i\frac{d\rho}{\rho}) : E^-_{\partial M} \longrightarrow E^0. \tag{1.12}$$

The combined identification is then

$$M : E_{\partial M} \longrightarrow E^0 \oplus E^0. \tag{1.13}$$

The indicial homomorphism for b-differential operators (discussed in [20]) defines, using ρ to trivialize the normal bundle to the boundary, the indicial family of $\eth$, $I(\eth, \lambda) \in \mathrm{Diff}^1_{\partial\phi}(\partial M; E)$. By the identification (1.13) and the boundary behaviour of the Levi-Civita connection this becomes

$$M \cdot I(\eth, \lambda) \cdot M^{-1} = \sigma\eth^0 + \gamma\lambda, \tag{1.14}$$

where (see [20]) $\eth^0$ is the Dirac operator on E^0 defined by the Clifford action (1.10) on E^0, and σ and γ are the 2×2 matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.15}$$

2. Spectral sections

For a single elliptic operator, in the usual sense, on a compact manifold with boundary there is an obstruction in K-theory to the existence of a local elliptic boundary condition [2]. This indeed underlies the global nature of the Atiyah-Patodi-Singer boundary condition. For a family of Dirac operators there is no analogous obstruction to the existence of a boundary condition of 'generalized Atiyah-Patodi-Singer' type, varying smoothly with the parameter. Boundary conditions in this sense are considered in the next three sections.

Let $\psi : M' \longrightarrow B$ be a fibration with compact fibres diffeomorphic to the manifold Y, $\partial Y = \emptyset$. Let $D \in \mathrm{Diff}^1_{\psi}(M'; E^0)$ be a family of self-adjoint elliptic operators on the fibres of ψ. In [7] Atiyah and Singer associate to D an element of $K^1(B)$ in two ways. First the principal symbol, $\sigma(D)$, of D defines an element $[\sigma(D)] \in K^1(T^*(M'/B))$ which, via the Thom isomorphism, fixes an element $\text{t-Ind}[\sigma(D)] \equiv \text{t-Ind}(D) \in$

$K^1(B)$. Secondly $K^1(B)$ is realized in [7] as the space of homotopy classes of maps from B into $\widehat{F}(H)$, the self-adjoint Fredholm operators on a separable Hilbert space H :

$$K^1(B) \cong [B, \widehat{F}(H)]. \tag{2.1}$$

A Hilbert bundle, such as that with fibre $L^2(M'_z; E^0)$ over $z \in B$, is always trivial, so D fixes an element a-Ind(D) by applying (2.1) to

$$Q = (D^2+1)^{-\frac{1}{2}} D \in \Psi^0_\psi(M'; E^0) \tag{2.2}$$

as a Fredholm family. The Atiyah-Singer index theorem becomes

$$\text{a-Ind}(D) = \text{t-Ind}(D). \tag{2.3}$$

We shall now give an alternative characterization of the vanishing of a-Ind(D) in terms of the existence of a spectral section in the following sense.

Definition 1. A spectral section for a family of self-adjoint elliptic operators $D \in \text{Diff}^1_\psi(M'; E^0)$ is a family of self-adjoint projections $P \in \Psi^0_\psi(M'; E^0)$ such that for some smooth function $R : B \longrightarrow \mathbb{R}$ (depending on P) and every $z \in B$

$$D_z u = \lambda u \Longrightarrow \begin{cases} P_z u = u \text{ if } \ \lambda > R(z), \\ P_z u = 0 \text{ if } \ \lambda < -R(z). \end{cases} \tag{2.4}$$

The definition can be extended to general families of self-adjoint Fredholm operators by dropping the differential and pseudodifferential requirements.

If (2.4) holds for $R(z)$ it holds for any larger function. In particular if B is compact $R(z)$ can be taken to be constant. Similarly, notice that $R(z) < 0$ in (2.4) implies that there are no eigenvalues of D_z in the interval $(R(z), -R(z))$; in general we shall take $R(z) \geq 0$. The most obvious case in which a spectral section exists is when the family has a spectral gap, i.e., there is a smooth function $\tau : B \longrightarrow \mathbb{R}$ such that $\tau(z)$ is not an eigenvalue of D_z for any $z \in B$. Then P_z can be taken to be orthogonal projection onto the span of the eigenspaces for D_z with eigenvalues greater than $\tau(z)$; (2.4) holds provided $R(z) > |\tau(z)|$. That the resulting family of projections is in $\Psi^0_\psi(M', E^0)$ follows from its representation as a contour integral in terms of the resolvent of D, as in the work of Seeley [23].

From now on we will assume the base B of the fibration to be compact.

Proposition 1. *For a family,* $D \in \operatorname{Diff}^1_\psi(M'; E^0)$, *of elliptic self-adjoint operators the existence of a spectral section is equivalent to the vanishing of* a-Ind(D) *in* $K^1(B)$.

Proof. Suppose first that the family has a spectral section and consider the normalized family Q in (2.2). Using the family of projections write

$$Q_z = P_z Q_z P_z + (\mathrm{Id} - P_z) Q_z (\mathrm{Id} - P_z) + Q'_z. \tag{2.5}$$

The condition (2.4) on P_z implies that Q'_z is a self-adjoint family of finite rank. Thus Q is homotopic to $Q - Q'$ in $\widehat{F}(H)$. It follows that, for any $r \in \mathbb{R}$, Q is homotopic in $\widehat{F}(H)$ to

$$\begin{aligned} \widetilde{Q}_z =& P_z(Q_z + rG)P_z + (\mathrm{Id} - P_z)(Q_z - rG)(\mathrm{Id} - P_z), \\ G =& (D^2 + \mathrm{Id})^{-\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Again using (2.4) the first term is strictly positive on the range of P_z and the second is strictly negative on its orthocomplement provided $r > 0$ is taken large enough. In particular $\widetilde{Q}_z$ is then invertible and hence its index vanishes, see [7]. q.e.d.

To prove the converse we start with the following simple result dealing with operators which are projections up to finite rank.

Lemma 1. *Let J be a bounded self-adjoint operator on a Hilbert space H with $J^2 - J$ of finite rank, and suppose that there is a self-adjoint projection Q with image H_Q, such that $\|Q - J\| < \frac{1}{2}$. Then $J : H_Q \longrightarrow H_P$ is an isomorphism onto a closed subspace of H; if P is the orthogonal projection onto H_P then $J - P$ has finite rank.*

Proof. For $u \in H_Q$ writing $Ju = Qu + (J - Q)u = u + (J - Q)u$ it follows that $\|Ju\| \geq \frac{1}{2}\|u\|$ so J is an isomorphism from H_Q to its image, H_P. Certainly for $u' \in H_P$, $u' = Ju$ with $u \in H_Q$ so $Ju' = J^2u = Ju + Fu$ where F has finite rank. Thus $J = \mathrm{Id} + F'$ on H_P with F' of finite rank. Consider the operator $T = \mathrm{Id} - J + JQ = \mathrm{Id} + F - J(J - Q)$. Our assumptions imply that $\|J(J - Q)\| < 1$ so $T - F$ is a bijection mapping the range of J into the sum of H_P and the range of F. Since F has finite rank it follows that the range of J is the sum $H_P \oplus G$ for some finite dimensional space G. Thus if $u' \perp H_P$ and $u' \perp G$ then $Ju' = 0$ by the assumed self-adjointness. Hence indeed $J - P$ is an operator of finite rank. q.e.d.

So now suppose that a-Ind$(D)=0$. The L^2 spaces on the fibres can be identified so that all operators act on a fixed Hilbert space, H. The family (2.2) is therefore homotopic to a constant through self-adjoint Fredholm families $Q_{z,t}$. In particular all operators in these families have discrete spectrum in some fixed open interval $(-\epsilon,\epsilon)$, $\epsilon>0$. Choose $\chi\in\mathcal{C}^\infty(\mathbb{R})$ with $\chi(\lambda)=0$ in $\lambda<0$ and $\chi(\lambda)=1$ in $\lambda>\frac{1}{2}\epsilon$. Using the spectral theorem consider the operators

$$J_{z,t}=\chi(Q_{z,t}). \tag{2.7}$$

These form a homotopy of bounded self-adjoint families, with $J_{z,0}$ constant in z. Moreover all elements are projections up to finite rank in the sense that $J^2_{z,t}-J_{z,t}=F_{z,t}$ is an operator of finite rank depending smoothly on t and z.

In fact for each t there is a smooth family of projections, $P_{z,t}$, with $P_{z,t}-J_{z,t}$ of finite rank. This is certainly the case for $t=0$. Moreover we can divide the t-interval into subintervals on which $\|J_{z,t}-J_{z,t'}\|<\frac{1}{2}$. Proceeding inductively suppose that $P_{z,t}$ exists for t the lower limit of such an interval. Then for any fixed t' in the interval apply Lemma 1 to $J'_{z,t'}=J_{z,t'}+P_{z,t}-J_{z,t}$ for each z. This is a projection up to finite rank, in the sense that $(J'_{z,t'})^2-J'_{z,t'}$ has finite rank and $\|J'_{z,t'}-P_{z,t}\|<\frac{1}{2}$. Thus $H'=J'_{z,t'}P_{z,t}H$ is a Hilbert bundle over B, and the orthogonal projection, $P_{z,t'}$, onto H' differs from $J'_{z,t'}$, and hence from $J_{z,t'}$, by an operator of finite rank. Therefore $P_{z,t}$ exists for each t.

Writing J_z for $J_{z,1}$ we have found a smooth family of projections P_z such that J_z-P_z has finite rank. We can further modify the projections so that $J_z-P_z=H_z$ has values in a finite span of eigenspaces of D_z. It can certainly be uniformly approximated by such finite rank operators, H'_z, simply by smoothly truncating the eigenvector expansion. Then P_z can be replaced by orthogonal projection onto the image of $(J_z-H'_z)P_z=(P_z+H_z-H'_z)P_z$. The resulting projection differs from J_z by an operator in the span of a finite number of eigenfunctions of Q_z for each z, and so defines a spectral section. Since $J\in\Psi^0_\psi(M',E^0)$ it follows that $P\in\Psi^0_\psi(M',E^0)$. q.e.d.

As an example consider the family of operators on $\mathbb{S}^1_\theta$ given by

$$D^+_z=-id/d\theta+z/2\pi,\quad z\in[0,2\pi]. \tag{2.8}$$

Let E^+ be the line bundle over $\mathbb{S}^1_\theta\times\mathbb{S}^1_z=M'$ with global sections the smooth functions over $\mathbb{R}^2$ satisfying

$$\begin{aligned} u(\theta+2\pi,z) = &\quad u(\theta,z), \\ \text{and } u(\theta,z+2\pi) = &\quad e^{-i\theta}u(\theta,z). \end{aligned} \tag{2.9}$$

Then D^+_z, acting on $E^+_z = E^+|_{\mathbb{S}^1_\theta\times\{z\}}$, defines a family of self-adjoint elliptic operators $D^+ \in \mathrm{Diff}^1_\psi(M';E^+)$ associated to the fibration

$$\psi : M' = \mathbb{S}^1_\theta \times \mathbb{S}^1_z \longrightarrow \mathbb{S}^1_z = B. \tag{2.10}$$

Since the eigenvalues of $D^+_z$are given by $\lambda_n(z) = n + z/2\pi$, we see that the spectral flow (see [5]) of the family D^+ is equal to one. It is known ([5]) that for a periodic one-parameter family Q of self-adjoint Fredholm operators, i.e. parametrized by the circle, the analytic index a-Ind$(Q) \in K^1(\mathbb{S}^1) = \mathbb{Z}$ is equal to the spectral flow. Thus our family D^+ has a non-trivial analytic index and hence cannot have a spectral section. Let E^- and D^- be obtained by changing the sign of z. Then $D_z = D^+_z \oplus D^-_z$ acting on $E^0 = E^+ \oplus E^-$ is self-adjoint and has vanishing spectral flow (and hence index); it has no spectral gap. Define P_z to be the identity on eigenspaces with eigenvalues 1 or greater, and 0 on eigenspaces with non-positive eigenvalues. This fixes P_z for $z = 0,\ 2\pi$ and for $z \in (0, 2\pi)$ off the eigenspaces corresponding to eigenvalues $z/2\pi$ for D^+_z and $1-z/2\pi$ for D^-_z. If u^+_z and u^-_z are orthonormal eigenfunctions for these eigenvalues, choose $\chi \in \mathcal{C}^\infty([0,2\pi])$ with $\chi(z) = 0$ in $z < 1/4$ and $\chi(z) = \pi/2$ in $z > 2\pi - 1/4$, and define P_z on the span of these two vectors as the othogonal projection onto the one dimensional subspace spanned by $w_z = a\cos\chi(z)u^-_z + b\sin\chi(z)u^+_z$. Then P_z is easily seen to be a spectral section.

Not only does the vanishing of the index of a family imply that it has a spectral section but there are then many such projections:

Proposition 2. *If the self-adjoint elliptic family $D \in \mathrm{Diff}^1_\psi(M';E^0)$ has a spectral section, then the set $\mathrm{SP}(D)$ of such spectral sections is infinite, and if $P, Q \in \mathrm{SP}(D)$ there exists $R, R' \in \mathrm{SP}(D)$ such that $R\cdot P = R$, $R\cdot Q = R$, $R'\cdot P = P$ and $R'\cdot Q = Q$. Given any $s \in \mathbb{R}$, sufficiently large, the spectral sections R and R' can be chosen to be the identity on all eigenfunctions of D corresponding to eigenvalues $\lambda > s$ and $\lambda > -s$ respectively.*

Proof. Let P be a spectral section for the family, and consider the family of operators PDP acting on the Hilbert bundle which is the range of P. These are self-adjoint operators and from (2.4) there is some integer N_1 such that all but the first N_1 eigenfunctions of PDP are eigenfunctions of D. For any fixed integer $N \geq N_1$ and some integer $M = M(N)$ we can find a smooth subbundle of the range of P which contains the first N eigenfunctions at each point and is contained in the span of the first M eigenfunctions. Let R be the orthogonal projection onto the orthocomplement of this subbundle extended to be zero on the range of $\mathrm{Id} - P$. Then R is a spectral section. If the integer N is chosen large enough, then the projection R will have range contained in the intersection of the ranges of any two given spectral sections P and Q. A similar argument using the range of $\mathrm{Id} - P$ allows the construction of a spectral section R'. The last part of the statement also follows by choosing N sufficiently large. q.e.d.

Suppose that P and R are two spectral sections for the family D such that $R \cdot P = R$. Then they define a $\mathcal{C}^\infty$ vector bundle over B with fibre at z

$$
\begin{aligned}
F_z(P,R) = & \quad \left\{ u \in \mathcal{C}^\infty(M'_z; E^0); \Pi_z u = u \right\}, \\
\Pi_z = & \quad P_z \cdot (\mathrm{Id} - R_z).
\end{aligned}
\tag{2.11}
$$

That these spaces form a smooth bundle over B follows directly from the fact that Π is a smooth family of finite rank projections over B. We shall later consider natural connections on these vector bundles.

Let us now assume that M' is equal to the boundary of a fibration M with compact fibres diffeomorphic to a fixed manifold with boundary X as in Section 1. Let $D = \eth^0$ be the boundary family associated to the given family of Dirac operators. In this case we have the following well-known result (stated already in [24]).

Proposition 3. *The analytic index of the boundary family $\eth^0$ is always zero in $K^1(B)$:*

$$
\text{a-Ind}(\eth^0) = 0. \tag{2.12}
$$

Proof. By the Atiyah-Singer family index theorem for self-adjoint families ([5]) it suffices to show that $\text{t-Ind}(\eth^0) \equiv \text{t-Ind}[\sigma(\eth^0)] = 0$ in $K^1(B)$. To see this consider the long exact sequence associated to the pair of locally compact spaces $(T^*(M/B), T^*_{\partial M}(M/B))$:

$$
K^0(T^*(M/B)) \xrightarrow{r} K^1(T^*(\partial M/B)) \xrightarrow{\delta} K^1(T^*(M/B), T^*_{\partial M}(M/B)),
$$

where r is obtained by restriction to the boundary followed by the identification

$$K^{-1}(T^*(\partial M/B)) \leftrightarrow K^1(T^*(\partial M/B))$$

given in [5] and δ is the connecting homomorphism. Trivializing the normal bundle to the boundary gives an inclusion of ∂M into the interior of the collar neighbourhood, thus fixing a map $j : \partial M \hookrightarrow \overset{\circ}{M}$. It follows from the definitions that $\delta = j_!$. Since the self-adjoint symbols defining the classes in $K^1(T^*(M/B), T^*_{\partial M}(M/B))$ are trivial when restricted to the boundary, there is a well defined topological index homomorphism

$$\text{t-Ind} : K^1(T^*(M/B), T^*_{\partial M}(M/B)) \longrightarrow K^1(B),$$

and it follows from the transitivity of the Thom homomorphism that

$$\text{t-Ind}(\alpha) = \text{t-Ind}(j_!\alpha) \quad \text{in } K^1(B)$$

for each $\alpha \in K^1(T^*(\partial M/B))$. Thus

$$\text{t-Ind}[\sigma(\eth^0)] = \text{t-Ind}(j_![\sigma(\eth^0)]) = \text{t-Ind}(j_! r[\sigma(\eth^+)]) = 0,$$

which, together with (2.3), proves the proposition. q.e.d.

Corollary 1. *Given a family of Dirac operators $\eth$ as in Section 1 there always exists a spectral section P for the boundary family $\eth^0$.*

3. Elliptic boundary problems

The metrically incomplete setting in which the Atiyah-Patodi-Singer boundary condition appears is geometrically the same as that discussed in Section 1 except that the family of complete metrics, (1.2), is replaced by a family of smooth metrics with product decomposition near the boundary. Let $\widehat{M}$ be the total space in this case. Thus for some decomposition of a neighbourhood $\Omega \equiv [0, \epsilon)_{x'} \times \partial\widehat{M}$, $0 < \epsilon < 1$, of the boundary the family of fibre metrics takes the form

$$g_{\widehat{M}/B} = (dx')^2 + h, \tag{3.1}$$

where h is a family of fibre metrics for the boundary fibration $\partial\widehat{M} \longrightarrow B$. With the b-tangent bundle replaced by the ordinary tangent bundle throughout the discussion in Section 1 still applies provided the superbundle E, now denoted $\widehat{E}$, has a product connection near the boundary. The Dirac operator, fixed by the same formula (1.6), becomes

$$M \circ \widehat{\eth} \circ M^{-1} = \gamma \frac{1}{i} \frac{\partial}{\partial x'} + \sigma \eth^0 \tag{3.2}$$

near the boundary, with $\eth^0$ the boundary Dirac operator for the bundle E^0.

If P is a spectral section for $\eth^0$ as introduced in Section 2 the boundary problem

$$\widehat{\eth}^+ u = f, \;\; P(u \restriction \partial \widehat{M}) = 0 \tag{3.3}$$

is an elliptic boundary problem varying smoothly with the base parameters. Let $\mathcal{L}^2(\widehat{M}; E^+)$ be the space of sections of E^+ which are square-integrable over each leaf and depend continuously on the base point. Thus, extended as a map on each leaf:

$$\begin{gathered} (\widehat{\eth}^+, P) : \mathrm{Dom}(\widehat{\eth}^+, P) \longrightarrow \mathcal{L}^2(\widehat{M}; \widehat{E}^-) \text{ with} \\ \mathrm{Dom}(\widehat{\eth}^+, P) = \Big\{ u \in \mathcal{L}^2(\widehat{M}; \widehat{E}^+); \widehat{\eth}^+ u \in \mathcal{L}^2(\widehat{M}; \widehat{E}^-), \\ P(u \restriction \partial \widehat{M}) = 0 \Big\} \end{gathered} \tag{3.4}$$

to the natural Sobolev spaces this defines a continuous family of Fredholm operators. Since the boundary conditions differ from the Atiyah-Patodi-Singer boundary conditions by finite rank operators this follows as in [4]; the crucial property is the continuity of the boundary conditions in the parameters. The virtual bundle formed by the difference of the null spaces of these operators and the null spaces of the adjoint family defines an element which we shall denote

$$\mathrm{Ind}(\widehat{\eth}, P) \in K^0(B). \tag{3.5}$$

The family Atiyah-Patodi-Singer index theorem proved below expresses the Chern character of this element of $K^0(B)$ in terms of characteristic classes and eta invariants. As in [4] the proof follows by adding a 'cylindrical end' to the total space of the fibration and passing to the complete space. This can be done explicitly by introducing as new coordinate $x = \exp(x')$, which transforms the collar $\Omega = [0, \epsilon)_{x'} \times \partial \widehat{M}$ to $[1, e^{\epsilon}) \times \partial \widehat{M}$. Attaching the infinite collar to the original manifold then amounts to extending the variable x down to 0, let M be the resulting compact manifold with boundary:

$$M = \widehat{M} \cup [0,1) \times \partial\widehat{M},\ \partial\widehat{M} \equiv \partial M. \tag{3.6}$$

Since $dx/x = dx'$ the original manifold, with metric, is identified with the region $x \geq 1$ in M, a compact manifold with boundary, with boundary defining function x and family of exact b-metrics as in (1.2). Note that M and $\widehat{M}$ are diffeomorphic as compact manifolds with boundary, but not in any natural way whereas their boundary are naturally diffeomorphic. The bundle $\widehat{E}$ and its connection can also be extended using the product trivialization near the boundary, giving a bundle E with b-connection induced by a true (product) connection. Let $\eth^+$ denote the Dirac operator for the extended manifold with b-metric. After discussing boundary conditions for the exact b-metrics we show, in Lemma 2, how the null space of the boundary problem (3.4) can be identified with the null space of $\eth^+$ acting on an appropriate domain on the fibres of M.

4. b-boundary problems

Next we discuss the assignment of boundary conditions of generalized Atiyah-Patodi-Singer type in the complete case. In general the 'principal symbol' of these boundary conditions is given by a spectral section of the boundary family. Consider first the case of a single operator on a fixed manifold with boundary X, and suppose that the metric has a product decomposition near the boundary, $g = (dx/x)^2 + h$ where h is the pull-back of a metric form ∂X. In this case the boundary condition, in the form of the selection of a domain, is quite direct.

Since the metric g is assumed to have a product decomposition near the boundary, the operator $\eth^+$ can be identified with its indicial operator in a collar neighbourhood of the boundary, i.e., in $x < \epsilon$, for some $\epsilon > 0$, $\eth^+$ corresponds to (1.14):

$$M \cdot \eth^+ \cdot M^{-1} = \eth^0 + x\frac{\partial}{\partial x}. \tag{4.1}$$

In this case any element of the formal null space

$$\operatorname{null}_{\mathrm{Fo}}(\eth^+) = \left\{ u \in \mathcal{C}^{-\infty}(X; E^+); \eth^+ u \in \dot{\mathcal{C}}^{\infty}(X; E^-) \right\}, \tag{4.2}$$

where $\mathcal{C}^{-\infty}$ denotes the space of all extendible-distributional sections, has a rather simple complete asymptotic expansion at the boundary of the form

$$u \sim \sum_{\lambda \in \operatorname{spec}(\eth^0)} x^{-\lambda} e_\lambda^+, \tag{4.3}$$

where $\eth^0 e_\lambda^+ = \lambda e_\lambda^+$; necessarily λ is bounded from above in the sum.

Let Φ_λ be the eigenspace of $\eth^0$ with eigenvalue λ, and let Σ_+^0 be the direct sum of the Φ_λ as $\lambda \to \infty$ and the direct product as $\lambda \to -\infty$. Thus elements of Σ_+^0 are sequences of eigenfunctions $\{e_\lambda^+\}$ with $e_\lambda^+ \in \Phi_\lambda$ and with $e_\lambda^+ = 0$ for λ sufficiently large and positive, but with the sequence unconstrained as $\lambda \to -\infty$. The asymptotic expansion (4.3) gives a map, b_+, into Σ_+^0, by setting $x = 1$. In fact the existence of asymptotic sums (É. Borel's lemma) shows that

$$0 \longrightarrow \dot{\mathcal{C}}^\infty(X; E^+) \hookrightarrow \operatorname{null}_{\mathrm{Fo}}(\eth^+) \xrightarrow{b_+} \Sigma_+^0 \longrightarrow 0 \tag{4.4}$$

is an exact sequence, where $\dot{\mathcal{C}}^\infty(X; E^+) \subset \mathcal{C}^\infty(X; E^+)$ is the space of sections vanishing to infinite order at the boundary. This sequence does not split linearly. We call b_+ a boundary map, of course one can get such a boundary map by evaluating the series at any other point $x = x_0 > 0$.

Let P be a spectral section, still for a single operator. Then P acts on Σ_+^0, and the boundary condition we impose on $\eth^+$ is

$$Pb^+ u = 0, \ \ u \in \operatorname{null}_{\mathrm{Fo}}(\eth^+). \tag{4.5}$$

Lemma 2. *If $\eth^+$ is the Dirac operator obtained by extension to a cylindrical end, as discussed in Section 3, then the solutions of the original boundary problem (3.4) are precisely the restrictions to $\widehat{M} \subset M$ of the solutions to $\eth^+ u = 0$ which are tempered and satisfy (4.5) for the boundary map arising from evaluation of (4.3) at $x = 1$.*

Proof. The Dirac operator takes the form (3.2), so any solution of $\widehat{\eth}^+ u = 0$ in the collar $[0, \epsilon) \times \partial\widehat{M}$ can be expanded in the eigenfunctions of $\eth^0$ in the form

$$u = \sum_{\lambda \in \operatorname{spec}(\eth^0)} a_\lambda e^{-\lambda x'} \phi_\lambda \tag{4.6}$$

with the coefficients, $a_\lambda e^{-\lambda x'}$, rapidly decreasing in λ for $x' \in [0, \epsilon)$. Each term extends to an element of the null space of $\eth^+$, and the boundary condition $P(u \restriction \{x' = 0\}) = 0$ ensures that there are only a finite number of exponentially increasing terms in $x' < 0$. Thus the series (5.5) converges rapidly in $x' < 0$, and the change of variable to $x = e^{x'}$ reduces it to the form (4.3) with (4.5) valid. This shows that any solution of the incomplete problem extends. The converse is even more obvious, since (4.5) just reduces to the boundary condition in (3.3). q.e.d.

The discussion for $\eth^-$ is the same with some sign reversal. Thus the expansion (4.3) is replaced by

$$u_- \sim \sum_{\lambda\in\operatorname{spec}(\eth^0)} x^\lambda e_\lambda, \tag{4.7}$$

where now λ is bounded below in the sum. Thus we take Σ^0_- to be the direct product of the Φ_λ as $\lambda \to \infty$ and the direct sum as $\lambda \to -\infty$. This gives an exact sequence as in (4.4) with all signs reversed. Notice that Σ^0_+ and Σ^0_- have a non-degenerate pairing, given by the L^2 inner product on the eigenspaces:

$$\Sigma^0_+ \times \Sigma^0_- \in (\{e^+\},\{e^-\}) \longmapsto \sum_{\lambda\in\operatorname{spec}(\eth^0)} \langle e^+_\lambda, e^-_\lambda\rangle. \tag{4.8}$$

We naturally take the boundary condition on $\eth^-$ to be the adjoint condition

$$(\operatorname{Id} - P)b^- v = 0,\ v \in \operatorname{null}_{\mathrm{Fo}}(\eth^-). \tag{4.9}$$

There is another description of these boundary conditions. Namely by a boundary condition for the Dirac operator we shall mean a subspace of the formal null space $G \subset \operatorname{null}_{\mathrm{Fo}}(\eth^+)/\dot{\mathcal{C}}^\infty(X;E^+)$ which lies between weights in the sense that

$$\begin{gathered}\exists\ s \in \mathbb{R} \text{ such that}\\ u \in x^s H^\infty_b(X;E^+),\ \eth^+ u \in \dot{\mathcal{C}}^\infty(X;E^-) \Longrightarrow [u] \in G,\\ u \in \mathcal{C}^{-\infty}(X;E^+), \eth^+ u \in \dot{\mathcal{C}}^\infty(X;E^-),\\ [u] \in G \Longrightarrow u \in x^{-s}H^\infty_b(X;E^+).\end{gathered} \tag{4.10}$$

Lemma 3. *Any boundary condition for the Dirac operator, in the sense of (4.10), is given by (4.5) for a unique spectral section and conversely.*

Proof. The conditions in (4.10) mean exactly that the image of G under the boundary map b_+, contains all the Φ_λ for $\lambda < -s$ and does not intersect the Φ_λ for $\lambda > s$. Thus if $\mathrm{Id} - P$ is the unique self-adjoint projection onto $b_+ G$, then P is a spectral section and conversely. q.e.d.

In Section 8 a general boundary condition of this type is reduced to the usual Atiyah-Patodi-Singer boundary condition, in this case just making the domain L^2. To do so the operator $\eth^+$ is perturbed to a b-pseudodifferential operator. To show that the K-class of the index bundle is not changed in the process we need to follow the perturbation with a smooth family of boundary conditions.

Let P be a spectral section for $\eth^0$. The perturbation in Section 8 replaces the indicial family, $\eth^0 + i\lambda$, of $\eth^+$ by

$$\eth^0 + i\lambda + \tau\widehat{\rho}_\epsilon(\lambda)A^0. \tag{4.11}$$

Here A^0 is a finite rank smoothing operator with kernel expressible as a quadratic form, with coefficients smooth on the base, in the eigenforms of $\eth^0$ with eigenvalues in some finite range $[-s, s]$. Working locally in the base we can assume that $\pm s$ are not eigenvalues of $\eth^0$. The matrix A^0 is such that

$$\begin{gathered}\eth^0 + A^0 \text{ is invertible with}\\ P \text{ given by projection onto}\\ \text{the positive eigenspaces.}\end{gathered} \tag{4.12}$$

The function $\widehat{\rho}_\epsilon(\lambda)$ is entire in λ. Indeed we choose $\widehat{\rho}_\epsilon(\lambda) = \widehat{\rho}(\lambda\epsilon)$ where $\widehat{\rho}$ is the Fourier transform of $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ which is real-valued, has integral 1 and is even. Thus $\widehat{\rho}_\epsilon(\lambda)$ is an even entire function which approaches 1 uniformly on compact sets as $\epsilon \downarrow 0$ and satisfies estimates

$$|\widehat{\rho}_\epsilon(\lambda)| \le C_N(1 + |\lambda|\epsilon)^{-N} \text{ in } |\operatorname{Im}\lambda| \le C/\epsilon. \tag{4.13}$$

Lemma 4. *For $\epsilon > 0$ sufficiently small the only singular values of the family (4.11), i.e., values of $\lambda \in \mathbb{C}$ for which the null space is non-trivial, in $|\operatorname{Im}\lambda| \le s$ are points at which the inverse of the family has a simple pole with finite rank residue; the sum of the ranges of these residues is equal to the sum of the eigenspaces of $\eth^0$ in $[-s, s]$, provided $\pm s$ are not eigenvalues of $\eth^0$. As $\epsilon \downarrow 0$ the singular values and the associated null spaces uniformly approach the eigenvalues and eigenspaces of $\eth^0 + \tau A^0$.*

Proof. The operator (4.11) is diagonal for the decomposition into the eigenspaces of $\eth^0$ with eigenvalues in $[-s, s]$ and the complement. On the latter $A^0 = 0$. Thus it suffices to consider the first, finite dimensional, part. The singular values are then just the zeros of the determinant. The estimate (4.13) shows that there can be no singular values in $|\operatorname{Im}\lambda| < C$ with large real part. As $\epsilon \downarrow 0$ the zeros of this determinant in $|\operatorname{Im}\lambda| \le s$ must approach eigenvalues of $\eth^0 + \tau A^0$. The fact that all singular values are algebraically simple, i.e., the inverse has only a simple pole, follows from the fact that the derivative in λ approaches $i\,\mathrm{Id}$ as $\epsilon \downarrow 0$. q.e.d.

For a family with indicial operator (4.11) a generalized boundary condition is given by the projection P for all values of $\tau \in [0,1]$ and all small ϵ. Consider

$$\widetilde{\eth}^+ = \eth^+ + \tau\phi(x)\widehat{\rho_\epsilon}(xD_x)A^0\phi(x). \tag{4.14}$$

Here $\widehat{\rho_\epsilon}(xD_x)$ is properly an operator on the infinite collar $[0,\infty)_x$, but is well-defined on the manifold M since the cut-off function $\phi(x)$ is assumed to have support in $x < 1$ and to be identically equal to 1 in $x < \frac{1}{2}$. Then, as shown in [20], any formal solution of $\widetilde{\eth}^+$ with growth at worst $O(x^{-s})$, i.e., any $u \in x^{-s}H_b^{-\infty}(M; E^+)$ which satisfies $\widetilde{\eth}^+ u \in \dot{\mathcal{C}}^\infty(M; E^-)$ has a complete asymptotic expansion just as in (4.7) with the sum over those λ in $\operatorname{Im}\lambda < s$ which are singular values of the family (4.11) with the e_λ elements of the null space at the point λ. The association of a boundary map b_+ to a projection which annihilates all the null spaces for singular values in $\operatorname{Im}\lambda < -s$ and acts as the identity on all null spaces for singular values in $\operatorname{Im}\lambda > s$ extends immediately. Thus the one projection, P, a spectral section for $\eth^0$ fixes boundary conditions for the whole family (4.14) for $\tau \in [0,1]$ and $\epsilon \in [0,\epsilon_0]$, $\epsilon_0 > 0$ sufficiently small. The assumption (4.12) means that at $\tau = 1$, for $\epsilon > 0$ sufficiently small, this boundary condition is just the requirement that the solution be square-integrable.

5. The non-product case

In this section we discuss the formulation of boundary conditions, corresponding to a choice of spectral section, without demanding that the exact b-metric be a strict product near the boundary.

The boundary condition (4.5), for elements of the formal null space, makes sense under the assumption that the metric is a product near the boundary. If we consider the general case of an exact b-metric, then the asymptotic expansion of elements of the formal null space can be somewhat more complicated than (4.3). In this case we use the construction of formal solutions, see [20, §5.19], to produce quantization maps

$$q_+ : \Sigma^0_+ \longrightarrow \mathrm{null}_{\mathrm{Fo}}(\eth^+)/\dot{\mathcal{C}}^\infty(X;E^+). \tag{5.1}$$

We can define such a map in terms of any local product decomposition near the boundary, $X \cong \partial X \times [0,\epsilon)_x$, $\epsilon > 0$, where x is a boundary defining function corresponding to the exact b-metric. Indeed the differential operator has Taylor series at the boundary

$$D = I(D) + \sum_{j=1}^{\infty} x^j D^{(j)}, \tag{5.2}$$

where $I(D)$ and the $D^{(j)}$ are $\mathbb{R}^+$-invariant operators in terms of this decomposition. For the Dirac operator associated to a product metric these $D^{(j)}$'s are all zero. As discussed in [20] the indicial operator $I(D)$ has a natural (formal) inverse on asymptotic expansions, given in terms of the Mellin transform. Using this we define q_+ recursively:

$$\begin{gathered} q_+(e_\lambda) \sim q_+^{(0)}(e_\lambda) + \sum_{j\ge 1} q_+^{(j)}(e_\lambda),\ q_+^{(0)}(e_\lambda) = x^{-\lambda} e_\lambda \\ q_+^{(j)}(e_\lambda) = -\sum_{p=1}^{\infty} I(D)^{-1}\left(x^p D^{(p)} q_+^{(j-1)}(e_\lambda)\right),\ j \ge 1. \end{gathered} \tag{5.3}$$

This has the property

$$q_+(e_\lambda) \sim x^{-\lambda} e_\lambda + \sum_{\substack{r\in \mathrm{spec}(\eth^0)-\mathbb{N}_0 \\ r\le \lambda-\delta}} \sum_{0\le j\le N(r)} x^{-r}(\log x)^j u_{j,r} \tag{5.4}$$

for some $\delta > 0$. Thus the error term, compared to the product case, is of strictly lower order than the leading part. In fact one can always arrange that $\delta = 1$.

Such a quantization map induces an isomorphism

$$q_+ : \Sigma^0_+ \longrightarrow \mathrm{null}_{\mathrm{Fo}}(\eth^+)/\dot{\mathcal{C}}^\infty(X;E^+),$$

the inverse will be written b_+ and called again a boundary map. There is in general no completely natural choice of such an isomorphism but note that, because of (5.4), if $q_+^{(i)}$ are quantization maps for $i = 1, 2$ with inverses $b_+^{(i)}$, then the difference has the property that

$$h^{(i)} = b_+^{(i)} \cdot (q_+^{(2)} - q_+^{(1)})$$

is a self-map of Σ_+^0 which is of negative order in the sense that

$$h^{(i)} : \Phi_\lambda \longrightarrow \bigoplus_{r \in \operatorname{spec}(\eth^0), r \le \lambda - \delta} \Phi_r. \tag{5.5}$$

In fact we have associated each quantization map to a choice of product decomposition with boundary defining function consistent with the metric. Any two quantization maps are therefore homotopic through quantization maps with $\delta > 0$ fixed.

Once a boundary map has been chosen we can impose a boundary condition on $\eth^+$ as in Section 4 by requiring that

$$Pb^+ u = 0, \ u \in \operatorname{null}_{\mathrm{Fo}}(\eth^+). \tag{5.6}$$

The alternative description (4.10) in terms of a subspace G of the formal null space can be given again and the following extension of Lemma (3) holds

Lemma 5. *Any boundary condition for the Dirac operator, in the sense of (4.6), is given by (4.10), with respect to a choice of boundary map, for a unique spectral section and conversely. For a fixed boundary condition different choices of boundary map give homotopic spectral sections.*

Proof. Once a boundary map associated to a product decomposition has been chosen, the argument in the proof of Lemma 3 giving the identification of the two boundary conditions applies unchanged. The homotopy arises from the fact that the boundary maps are themselves homotopic under changes of product decomposition and admissible boundary defining function. q.e.d.

To discuss the boundary condition for $\eth^-$ we point out that the non-degenerate pairing (4.8) arises from a pairing of the formal null spaces, which persists in the non-product case. This is discussed in [20, Chapter 6]. The pairing is defined by

$$\begin{aligned} \langle u, v \rangle &= \langle \eth^+ u, v \rangle - \langle u, \eth^- v \rangle \\ &= \int_X \left(\eth^+ u \bar{v} - u \overline{\eth^- v} \right) dg. \end{aligned} \tag{5.7}$$

It is always non-degenerate (see [20, Chapter 6]). We use it to relate quantization maps for $\eth^+$ and $\eth^-$. Thus q_+ and q_- are dual quantization maps if in terms of these inner products

$$\langle q_+ e, v\rangle = \langle e, b_- v\rangle,\ b_- = q_-^{-1}. \tag{5.8}$$

This identifies the boundary map for $\eth^+$ which is the adjoint of a given boundary map for $\eth^-$ and the boundary condition for $\eth^-$ is

$$(\mathrm{Id} - P)b^- v = 0,\ v \in \mathrm{null}_{\mathrm{Fo}}(\eth^-). \tag{5.9}$$

In summary by a boundary condition of generalized Atiyah-Patodi-Singer type on the Dirac operator $\eth$ on an exact b-manifold we shall mean the conditions (5.6) and (5.9) on the formal null space of $\eth^+$ and $\eth^-$ where P is a spectral section of the boundary operator and $b_\pm$ are adjoint boundary maps.

6. Index bundle

We now turn to families of Dirac operators, the first question being smoothness in the parameter space. Changes of multiplicity mean that the eigenspaces, $\Phi_\lambda(z)$, do not really vary smoothly with z. However there is a natural local bundle structure on the spaces

$$\sum_{|\lambda-\lambda'|<\epsilon} \Phi_\lambda(z) \tag{6.1}$$

for $\epsilon > 0$ so small that λ' is the only eigenvalue of $\eth^0_{z'}$ in the interval $[\lambda' - \epsilon, \lambda' + \epsilon]$ and then for z in a sufficiently small neighbourhood of $z' \in B$. Namely if e_j for $j = 1, \dots, m(\lambda', z')$ are a basis of $\Phi_{\lambda'}(z')$, then the contour integrals

$$\oint_{|\zeta-\lambda'|=\epsilon} (\eth^0_z - \zeta)^{-1} e_j \, d\zeta$$

give a basis of (6.1) varying smoothly with z. In general it is not a basis of eigenvectors. Since this notion of smoothness is independent of the choices made it shows that the $\Sigma^0_+(z)$ and $\Sigma^0_-(z)$, associated to $\eth^0_z$ for each $z \in B$, form $\mathcal{C}^\infty$ bundles over B, at least in the sense that we have defined the space of smooth sections. By assumption a spectral section is a smooth family of pseudodifferential operators so it preserves the smoothness of sections of these bundles.

This smooth structure on the bundles $\Sigma^0_\pm$ induces in a natural way a notion of smoothness for sections of the quotient spaces $\mathrm{null}_{\mathrm{Fo}}(\eth^+)(z)/\dot{\mathcal{C}}^\infty(M_z;E^+)$. Namely if the product decomposition is chosen to be smooth, then the resulting boundary maps $b_+(z)$ on each fibre will be considered smooth in z. This notion of smoothness is again independent of the choices made. Thus a smooth boundary condition of generalized Atiyah-Patodi-Singer type for a family $\eth$ is a choice of smoothly varying subspaces $G_z \in \mathrm{null}_{\mathrm{Fo}}(\eth^+)(z)/\dot{\mathcal{C}}^\infty(M_z;E^+)$ satisfying (4.10) for each z, with s smooth in z, corresponding to a spectral section for the family $\eth^0$ under one (hence any) family of admissible boundary maps. By Corollary 1 such a boundary condition always exists.

In [6] (see also [1, Appendix]) Atiyah and Singer show that the kernel and cokernel of a continuous family of Fredholm operators always define a virtual bundle, i.e., an element of the K-theory of the parameter space. Here we need to extend this construction slightly to the case where there is everywhere locally a Fredholm family with the given kernel and cokernel:

$$\begin{aligned} &\mathrm{null}(\eth^+,P)(z) = \left\{u \in \mathcal{C}^{-\infty}(M_z;E^+); \eth^+u = 0, Pb_+u = 0\right\}, \\ (6.2)\quad & \\ &\mathrm{null}(\eth^-,P)(z) = \left\{u \in \mathcal{C}^{-\infty}(M_z;E^-); \eth^-u = 0, (\mathrm{Id}-P)b_-u = 0\right\}. \end{aligned}$$

The families we define do not, in general, extend globally. In Section 8 we show how to deform the virtual bundle to one which is defined globally by a Fredholm family.

Consider the setting of Section 4, where $b_\pm$ are dual boundary maps for $\eth^\pm$, and P is a spectral section for the boundary family. We proceed to define Fredholm families associated to these data. Consider an open set $U \subset B$ and $s \in \mathbb{R}$ such that $s > R(z)$ and $\eth^0_z + s$ is invertible for all $z \in U$. Here $R(z)$ is the function in (2.4). Define

$$\begin{aligned} H_{s,1,+}(z) =& x^s H^1_b(M_z;E^+) \\ &+ \left\{u \in \mathrm{null}_{\mathrm{Fo}}(\eth^+); Pb_+u = 0\right\},\ \ z \in U. \end{aligned} \tag{6.3}$$

The boundary map projects to

$$\begin{aligned} &b_+(s) : H_{s,1,+}(z) \longrightarrow D_+(s,z), \\ &D_+(s,z) = \left\{ v \in \bigoplus_{-s<\lambda<R(z)} \Phi_\lambda; Pv = 0 \right\}. \end{aligned} \tag{6.4}$$

Indeed $b_+(s)$ on the second summand in (6.3) is just the truncation of b_+ obtained by dropping the components corresponding to the eigenvalues less than $-s$; it therefore vanishes on the intersection of the two summands and so can be extended by demanding that it vanish on $x^sH^1_b(M_z,E^+)$. In fact $b_+(s)$, so defined, vanishes precisely on $x^sH^1_b(M_z,E^+)\subset H_{s,1,+}(z)$ and hence defines a short exact sequence

$$\begin{aligned}0\longrightarrow x^sH^1_b(M_z,E^+)\longrightarrow H_{s,1,+}(z)&\\ \xrightarrow{b_+(s)} D_+(s,z)\longrightarrow 0.&\end{aligned}\tag{6.5}$$

The dimension of $D_+(s,z)$ is constant and it inherits a Euclidean structure from the L^2 inner product on E^+ over the boundary. We choose a linear quantization map

$$q_+(s): D_+(s,z)\longrightarrow H_{s,1,+}(z),\ b_+(s)\cdot q_+(s)=\mathrm{Id}\tag{6.6}$$

and declare $q_+(s)$ to be an isometry, so give $H_{s,1,+}(z)$ the unique Hilbert space structure for which (6.5) splits orthogonally:

$$H_{s,1,+}(z)=x^sH^1_b(M_z;E^+)\oplus q_+(s)\left(D_+(s,z)\right).\tag{6.7}$$

There are completely analogous constructions for the adjoint operator, under the assumption that $s>R(z)$ and $\eth^0-s$ is invertible for all $z\in U$

$$\begin{aligned}H_{s,1,-}(z)=&x^sH^1_b(M_z;E^-)\\ &+\left\{u\in \mathrm{null}_{\mathrm{Fo}}(\eth^-);(\mathrm{Id}-P)b_-u=0\right\},\\ &\qquad z\in U\end{aligned}\tag{6.8}$$

with a truncated boundary map

$$\begin{aligned}b_-(s): H_{s,1,-}(z)\longrightarrow D_-(s,z),\\ D_-(s,z)=\left\{v\in \textstyle\bigoplus_{s>\lambda>-R(z)}\Phi_\lambda;(\mathrm{Id}-P)v=0\right\}.\end{aligned}\tag{6.9}$$

By definition $D_+(s,z)\subset\Sigma^0_+(z)$ and $D_-(s,z)\subset\Sigma^0_-(z)$ are orthogonal with respect to the pairing (4.8) and

$$D_-(s,z)\oplus D_+(s,z)=\bigoplus_{-s<\lambda<s}\Phi_\lambda.\tag{6.10}$$

We can therefore choose the quantization map

$$q_-(s): D_-(s,z)\longrightarrow H_{s,1,-}(z),\ b_-(s)\cdot q_-(s)=\mathrm{Id}\tag{6.11}$$

such that

$$\begin{aligned}\langle \eth^+ q_+(s)u, q_-(s)v\rangle &= \langle q_+(s)u, \eth^- q_-(s)v\rangle \\ \forall\ u \in D_+(s,z),\ & v\in D_-(s,z)\end{aligned} \tag{6.12}$$

in terms of the L^2 pairing. With such a choice of quantization map there is a splitting, as in (6.7),

$$H_{s,1,-}(z) = x^s H^1_b(M_z; E^-) \oplus q_-(s)\left(D_-(s,z)\right). \tag{6.13}$$

In either case these spaces decrease with increasing s :

$$H_{s',1,\pm}(z) \subset H_{s,1,\pm}(z),\ s' > s, \tag{6.14}$$

provided both are defined, and there are natural (dense) inclusions:

$$H_{s,1,\pm}(z) \subset x^{-s} H^1_b(M_z; E^{\pm}). \tag{6.15}$$

Lemma 6. *If $U \subset B$ is an open set and $s \in \mathbb{R}$ is such that $s > R(z)$ and both $\eth^0_z + s$ and $\eth^0_z - s$ are invertible for all $z \in U$, then*

$$\begin{aligned}\eth^+_P &: H_{s,1,+}(z) \longrightarrow x^s H^0_b(M_z, E^-),\\ \eth^-_P &: H_{s,1,-}(z) \longrightarrow x^s H^0_b(M_z, E^+),\end{aligned} \tag{6.16}$$

defined by the action of $\eth^{\pm}$ on these Hilbert spaces, are smooth families of Fredholm operators with null spaces independent of s, given by (6.2) and with ranges the annihilators of the other's null space under the inclusion (6.15).

Proof. Basic results on the Fredholm properties of elliptic b-differential operators ([21], [20]) show that under the assumption that $\eth^0_z \pm s$ is invertible,

$$\eth^{\pm} : x^s H^1_b(M_z; E^{\pm}) \longrightarrow x^s H^0_b(M_z, E^{\mp}) \tag{6.17}$$

is Fredholm. Since $\eth^+$ maps the second summand in (6.3) into $\dot{\mathcal{C}}^\infty(M_z; E^-)$, and similarly for $\eth^-$, the operators in (6.16) are well defined. Moreover they are finite extensions of (6.17), so are Fredholm and clearly smooth in z. The independence of the null space of the parameter s follows from the identifications with the null spaces in (6.2).

To see these identifications note that an element of the first of the spaces in (6.2) is contained in the second summand in (6.3) and is certainly in the null space of the operator. Conversely the action of $\eth^+$

is consistent with the inclusion (6.15), so any element of the null space is contained in $\mathrm{null}_{\mathrm{Fo}}(\eth^+)$ and satisfies the generalized Atiyah-Patodi-Singer boundary condition for the spectral section P.

It remains to identify the ranges of these Fredholm operators. Consider the adjoint of $\eth^+_P$ as a map

$$\begin{aligned}(\eth^+_P)^* : x^{-s}H^0_b(M_z, E^-) \\ \longrightarrow x^{-s}H^{-1}_b(M_z; E^+) \oplus q_+(s)\,(D_+(s,z))\,.\end{aligned} \tag{6.18}$$

This is just the sum of the adjoints of $\eth^+$ as a map from each of the two summands in (6.7), using the L^2 pairings of the weighted Sobolev spaces. Thus the first term in (6.18) is just $\eth^-$. By definition the splitting (6.7) corresponds to the right inverse, $q_+(s)$, of $b_+(s)$ so the second part of (6.18) is determined by:

$$\langle b^+(s)(\eth^+_P)^*v, w\rangle = \langle v, \eth^+_P q_+(s)w\rangle \quad \forall\ w \in D_+(s,z). \tag{6.19}$$

By definition the pairing on the left is the L^2 inner product from the boundary and that on the right is the continuous extension of the L^2 pairing on M_z. Thus an element of the null space of (6.18) is an element $v \in x^{-s}H^0_b(M_z; E^-)$ such that $\eth^- v = 0$ and

$$\langle v, \eth^+_P q_+(s)w\rangle = 0 \quad \forall\ w \in D_+(s,z). \tag{6.20}$$

Since

$$\langle \eth^- v, g\rangle - \langle v, \eth^+ g\rangle = \langle b_- v, b_+ g\rangle, \tag{6.21}$$

for elements of the formal null spaces (see (5.7) (5.8)), it follows that

$$\langle b_- v, w\rangle = 0 \quad \forall\ w \in D_+(s,z). \tag{6.22}$$

Thus $b_-(s)v \in D_-(s,z)$. This identifies an element of the null space of (6.18) with an element of the null space of $\eth^-_P$. The map is obviously injective and the argument can be reversed, completing the proof of the proposition. q.e.d.

In view of this result if, for $z \in U$, the dimensions of $\mathrm{null}(\eth^+_{P,z})$ and $\mathrm{null}(\eth^-_{P,z})$ do not vary, then these vector spaces define smooth vector bundles over U. The proof given in [1] or [6] only covers continuous families of Fredholm operators but the same argument gives smoothness whenever the family is assumed to be smooth (see also [9, §5]). For example if for each $z \in U$ the operator $\eth^+_{P,z}$ is surjective, then the dimension of $\operatorname{null}(\eth^+_{P,z})$ does not vary (since it is equal to the index of the operator) and we obtain a smooth vector bundle over U.

We shall apply Lemma 6 to show that the formal difference of the null spaces in (6.2) defines an element of $K^0(B)$. This is a standard procedure which applies to any continuous family of Fredholm operators, as in [6].

Proposition 4. *If P is a spectral section for $\eth^0$ and $s_0 > R(z)$ is given, then there is a trivial bundle, i.e., injective linear map depending smoothly on z, with values in the sections having compact support in the interior,*

$$G_z : \mathbb{C}^N \longrightarrow \mathcal{C}^\infty_c(\mathring{M}_z; E^-), \tag{6.23}$$

which complements the range of $\eth^+_P$ for any $s < s_0$ which is admissible, i.e.,

$$\eth^+_{P,z} \oplus G_z : H_{s,1,+}(z) \oplus \mathbb{C}^N \longrightarrow x^s H^0_b(M_z; E^-) \tag{6.24}$$

is surjective for every $z \in B$ and $R(z) < s < s_0$ with $\eth^0 + s$ invertible.

Proof. We simply choose the trivial bundle defined by G_z to have range always including a complement to the range of $\eth^+_{P,z}$. To do so the range of G_z need only have pairing of maximal rank with the null space of $\eth^-_{P,z}$ under the duality between $x^s H^0_b(M_z; E^-)$ and $x^{-s} H^0_b(M_z; E^-)$. This can certainly be arranged for any fixed value $z_0 \in U$. Any basis of the null space of $\eth^-_{P,z_0}$, and thus of the range of G_{z_0}, can be approximated uniformly in the L^2-norm by smooth sections with compact support in the interior. Since the condition that $\eth^+_{P,z_0} \oplus G_{z_0}$ be surjective is an open one, we conclude that it is indeed possible to find G_{z_0} such that (6.23) and (6.24) hold for the fixed value $z_0 \in U$. A standard argument from [1] then shows that

$$\eth^+_{P,z} \oplus G_{z_0} : H_{s,1,+}(z) \oplus \mathbb{C}^N \longrightarrow x^s H^0_b(M_z; E^-) \tag{6.25}$$

is still surjective for z in an open neighbourhood of z_0. From the compactness of B and a partition of unity the result follows. q.e.d.

By Lemma 6 the null spaces of $\eth^+_{P,z} \oplus G_z$ are independent of the chosen s for each z in an open set on which s is admissible. Thus to every finite covering $\{U_i\}$ of B by open sets on which $s_i < s_0$ is admissible we can associate a local smooth family of surjective Fredholm operators

$(\eth_P^+ \oplus G)_{\mathcal{U}}$ with $\mathcal{U} = \{(U_i, s_i)\}$. The null spaces of such a family always define a smooth vector bundle $\operatorname{null}(\eth_P^+ \oplus G)_{\mathcal{U}}$, and it is clear from the remark just made that

$$\operatorname{null}(\eth_P^+ \oplus G)_{\mathcal{U}} = \operatorname{null}(\eth_P^+ \oplus G)_{\mathcal{V}}$$

for a different choice of $\mathcal{V} = \{(V_j, s_j)\}$ as above. We will denote this bundle by $\operatorname{null}(\eth_P^+ \oplus G)$.

Definition 2. Let P be a spectral section for the boundary family $\eth^0$. The formal difference of null spaces in (6.2) defines a class $\operatorname{Ind}_P(\eth) \in K^0(B)$ as follows

$$\operatorname{Ind}_P(\eth) = [\operatorname{null}(\eth_P^+ \oplus G)] - [B \times \mathbb{C}^N]. \tag{6.26}$$

The class (6.26) does not depend on the choice of the family of linear maps G, since if

$$\overline{G_z} : \mathbb{C}^{\overline{N}} \longrightarrow \mathcal{C}_c^\infty(\mathring{M}_z; E^-)$$

is another such family, then there are exact sequences of vector bundles

$$\begin{aligned} 0 \hookrightarrow \operatorname{null}(\eth_P^+ \oplus G) \hookrightarrow \operatorname{null}(\eth_P^+ \oplus G \oplus \overline{G}) \longrightarrow (B \times \mathbb{C}^{\overline{N}}) \longrightarrow 0, \\ 0 \hookrightarrow \operatorname{null}(\eth_P^+ \oplus \overline{G}) \hookrightarrow \operatorname{null}(\eth_P^+ \oplus G \oplus \overline{G}) \longrightarrow (B \times \mathbb{C}^N) \longrightarrow 0, \end{aligned}$$

so

$$[\operatorname{null}(\eth_P^+ \oplus G)] - [(B \times \mathbb{C}^N)] = [\operatorname{null}(\eth_P^+ \oplus \overline{G})] - [(B \times \mathbb{C}^{\overline{N}})]$$

as required. By homotopy invariance, the index bundle is also independent of the choice of boundary maps used in the definition of the boundary condition.

7. Relative index theorem

If P and Q are both spectral sections for the family $\eth^0$, then their difference defines a K-class on the parameter space:

$$[\operatorname{ran}(P) - \operatorname{ran}(Q)] \in K^0(B). \tag{7.1}$$

This needs to be justified, since the ranges of P and Q are both infinite dimensional. If $P \cdot Q = Q$, so the range of Q is contained in the range of P, then the difference bundle should be taken as the orthocomplement of $\operatorname{ran}(Q)$ in $\operatorname{ran}(P)$. This is the range of the projection $(\operatorname{Id} - Q) \cdot P$ on

$\operatorname{ran}(P)$ so is certainly a smooth subbundle. In the general case we can use Proposition 2 and choose a third spectral projection R such that $R \cdot P = R$ and $R \cdot Q = R$ and set

$$[P - Q] = [P - R] - [Q - R] \in K^0(B). \tag{7.2}$$

Lemma 7. *The class in (7.2) is independent of the choice of auxiliary spectral section R.*

Proof. If R' is another spectral section with $R' \cdot P = R'$ and $R' \cdot Q = R'$, i.e., with range contained in the intersection of the ranges of P and Q, then again by Proposition 2 there is a spectral section T with range contained in the intersection of the ranges of R and R'. It follows that if the various orthocomplements are defined by

$$\begin{aligned}
\operatorname{ran}(P) &= \operatorname{ran}(R) \oplus A,\ \operatorname{ran}(Q) = \operatorname{ran}(R) \oplus B,\\
\operatorname{ran}(P) &= \operatorname{ran}(R') \oplus A',\ \operatorname{ran}(Q) = \operatorname{ran}(R') \oplus B',\\
\operatorname{ran}(R) &= \operatorname{ran}(T) \oplus C,\ \operatorname{ran}(R') = \operatorname{ran}(T) \oplus C',
\end{aligned}$$

then

$$A \oplus C = A' \oplus C',\ B \oplus C = B' \oplus C',$$

which shows that $A - B$ and $A' - B'$ do indeed define the same K-class.

q.e.d.

In Section 6 we have defined the class of the index bundle for the generalized Atiyah-Patodi-Singer boundary problem associated to a spectral section P for the boundary family. Next we shall prove the relative index theorem which relates the index classes of two spectral sections. For the Chern character this result can also be deduced directly from the family index formula (2) and the main result in Section 16, but is much more elementary, so we give a direct proof.

Proposition 5. *If P and Q are spectral sections for the boundary family, then*

$$\operatorname{Ind}_Q(\eth) - \operatorname{Ind}_P(\eth) = [P - Q] \in K^0(B). \tag{7.3}$$

Proof. By Lemma 7 and Proposition 2 it suffices to prove (7.3) under the assumption that $P \cdot Q = Q$, so the range of Q is contained in the range of P. Consider the families of operators, D_P and D_Q, defined by (6.24) for the two boundary conditions. Provided s and N are large enough these are both surjective and have null spaces independent of s. Thus we can write the difference of the index bundles in the form

$$\operatorname{Ind}_Q(\eth) - \operatorname{Ind}_P(\eth) = \operatorname{null}(D_Q) \ominus \operatorname{null}(D_P). \tag{7.4}$$

The choice of a boundary map gives a sequence

$$\begin{aligned} 0 \hookrightarrow \operatorname{null}(D_P) &\hookrightarrow \operatorname{null}(D_Q) \\ &\xrightarrow{b_+} \mathcal{C}^\infty(B; [P-Q]) \longrightarrow 0. \end{aligned} \tag{7.5}$$

The exactness of this sequence reduces to the surjectivity of the boundary map. Let v be a smooth section of the quotient $[P-Q]$. Certainly there exists $u_1 \in \operatorname{null}_{\mathrm{Fo}}(\eth)$, the formal null space, with $b_+(u_1) = v$. From the definition of the bundle $[P-Q]$ it follows that $U_1 = (u_1, 0)$ is in the domain of the modified operator D_Q; clearly $D_Q U_1 = g \in \dot{\mathcal{C}}^\infty(M; E^-)$. By the surjectivity of D_P there exists U_2 with $D_P U_2 = g$. Then $U_1 - U_2 \in \operatorname{null}(D_Q)$ has v as its boundary data modulo $\operatorname{ran}(Q)$. Thus (7.5) is exact and the Proposition follows.

8. Regularization

In the regularization of the index bundle it is very useful to replace the given family of Dirac operators by one which has an invertible boundary family. As a first step towards this we use Proposition 1 to show that a self-adjoint family with trivial index has a finite rank homotopy to an invertible family. It is in this sense that we consider a spectral section as fixing an explicit trivialization of the K^1-class associated to the given self-adjoint family.

Lemma 8. *If $D \in \operatorname{Diff}^1_\psi(M'; E^0)$ has a spectral section P, over a compact base, then there is a family of self-adjoint smoothing operators $A^0_P \in \Psi^{-\infty}_\psi(M', E^0)$ with range in a finite sum of eigenspaces of D such that*

$$\widetilde{D} = D + A^0_P \in \Psi^1_\psi(M'; E^0) \tag{8.1}$$

is invertible and P is the Atiyah-Patodi-Singer projection onto the positive part of the spectrum of $\widetilde{D}$.

Proof. For the given spectral section, P, of D choose $0 < s \in \mathbb{R}$ such that P acts as the identity on all eigenfunctions of D with eigenvalues greater than or equal to s and annihilates all eigenfunctions corresponding to eigenvalues less than or equal to $-s$. Using Proposition 2 choose a second spectral section Q which annihilates all eigenfunctions corresponding to eigenvalues less than or equal to s. Again using Proposition 2 choose a third spectral section R which acts as the identity on all eigenfunctions corresponding to eigenvalues greater than or equal to $-s$. It follows that

$$PQ = QP = Q,\ PR = RP = P,\ QR = RQ = Q. \tag{8.2}$$

Thus the four operators Q, $\mathrm{Id}-R$, $PR(\mathrm{Id}-Q)$ and $(\mathrm{Id}-P)R(\mathrm{Id}-Q)$ are commuting self-adjoint projections, the second two having finite rank ranges. Moreover

$$R(\mathrm{Id}-Q) = PR(\mathrm{Id}-Q) + (\mathrm{Id}-P)R(\mathrm{Id}-Q)$$

is the orthogonal projection onto the orthocomplement of the sum of the ranges of Q and $(\mathrm{Id}-R)$. Thus the ranges of these four projections give an orthogonal decomposition of L^2. Consider the following operator which is block diagonal for this decomposition of L^2 :

$$\begin{aligned}\widetilde{D} =& Q\circ D\circ Q + sPR(\mathrm{Id}-Q)\\ &+ (\mathrm{Id}-R)\circ D\circ(\mathrm{Id}-R)\\ &- s(\mathrm{Id}-P)R(\mathrm{Id}-Q).\end{aligned} \tag{8.3}$$

From the properties of P and R the first two terms are strictly positive on the ranges of Q and $PR(\mathrm{Id}-Q)$ respectively, and similarly the last two terms are strictly negative on the ranges of $\mathrm{Id}-R$ and $(\mathrm{Id}-P)R(\mathrm{Id}-Q)$. Thus $\widetilde{D}$ is invertible. Defining $A^0_P = \widetilde{D} - D$ gives (8.1), and this self-adjoint operator has range in a finite span of eigenfunctions of D, since P, Q and R are spectral sections of D. From (8.2) it follows that

$$\begin{aligned}\widetilde{D}P =& P\widetilde{D} = Q\circ D\circ Q + sPR(\mathrm{Id}-Q),\\ \widetilde{D}(\mathrm{Id}-P) =& (\mathrm{Id}-P)\widetilde{D}\\ =& (\mathrm{Id}-R)\circ D\circ(\mathrm{Id}-R)\\ &- s(\mathrm{Id}-P)R(\mathrm{Id}-Q)\end{aligned} \tag{8.4}$$

are respectively the positive and negative parts of $\widetilde{D}$. Thus P is the Atiyah-Patodi-Singer projection for $\widetilde{D}$. q.e.d.

Corollary 2. *If D, P and A_P^0 are as in Lemma 8, then P is a spectral section of $\widetilde{D}_\tau = D + \tau A_P^0$ for all $\tau \in \mathbb{R}$ and is equal to the Atiyah-Patodi-Singer projection for $\widetilde{D} = \widetilde{D}_1$.*

When applied to the family of boundary Dirac operators, the resulting homotopy is within the space of pseudodifferential operators. However it does not give, directly, a suitable homotopy of operators in the interior, since the resulting family will be 'differential-pseudodifferential' and hence not pseudodifferential in the strict sense. Treating such a family complicates the analysis unnecessarily, so instead we further refine the homotopy.

Lemma 9. *Let P be a spectral section for the boundary family of a family of Dirac operators, $\eth$. Then on the fibres of $\phi : M \longrightarrow B$ there is a family $\widetilde{\eth}^+ \in \Psi^1_{b,\phi}(M; E^+, E^-)$ for which 0 is never the imaginary part of an indicial root and such that if $A_P^+ = \widetilde{\eth}^+ - \eth^+ \in \Psi^{-\infty}_{b,\phi}(M; E^+, E^-))$ then $M_- \circ I(A_P^+, z) \circ M_+^{-1}$ has kernel spanned by the eigenfunctions of $\eth^0$ for eigenvalues in a fixed range $[-R, R]$.*

Proof. We first construct the indicial family of the operator A_P^+. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ be non-negative, even and have integral 1, so $\rho_\epsilon(t) = \epsilon^{-1}\rho(t/\epsilon)$ approximates $\delta(t)$ as $\epsilon \downarrow 0$. The Fourier-Laplace transform

$$\widehat{\rho_\epsilon}(z) = \int e^{-itz}\rho_\epsilon(t)dt$$

is therefore entire, even, real for real z and has $\widehat{\rho_\epsilon}(0) = 1$. Then consider

$$I(A_P^+, z) = M_-^{-1} \circ \widehat{\rho_\epsilon}(z) A_P^0 \circ M_+ \tag{8.5}$$

where A_P^0 is the finite rank operator from Lemma 8, and $\epsilon > 0$ will be chosen small.

Certainly the kernel is spanned by a fixed range of the eigenfunctions of $\eth^0$. Consider the operator $I(\eth^+, z) + I(A_P^+, z)$, where the first term is given by (1.14). Reducing this to an operator on E^0 gives

$$iz + \eth^0 + \widehat{\rho_\epsilon}(z) A_P^0. \tag{8.6}$$

If z is real this operator is invertible. For $z \neq 0$ this follows from the self-adjointness of the second part. By construction $\eth^0 + A_P^0$ is invertible, so 0 is not an indicial root; thus 0 is not the imaginary part of any indicial root.

To construct the full operator A_P^+ we simply choose a product decomposition near the boundary; if $(A_P^+)'$ is the unique $\mathbb{R}^+$-invariant operator with indicial family $I(A_P^+, z)$ simply choose

$$A_P^+ = f(x)(A_P^+)' f(x'), \tag{8.7}$$

where $f(x)$ localizes near the boundary. q.e.d.

As $\epsilon \downarrow 0$ in (8.5) the factor $\widehat{\rho}_\epsilon(z)$ approaches 1 uniformly on any compact subset of $\mathbb{C}$ and is rapidly decreasing as $|\operatorname{Re} z| \to \infty$ provided $|\operatorname{Im} z|$ stays bounded. Choosing $\epsilon > 0$ small enough allows us to arrange that the indicial roots are very close to the eigenvalues of $\eth^0 + A$ in any preassigned strip $|\operatorname{Im} z| < N$ as shown in Lemma 4. This in turn allows smoothly varying boundary conditions to be specified for the homotopy

$$\widetilde{\eth}^{+,\tau} = \eth^+ + \tau A_P^+ \tag{8.8}$$

by the fixed projection P as described at the end of Section 4.

Finally we modify the family of operators $\widetilde{\eth}^+$ further by adding a family of finite rank operators, as in Section 6 to give a surjective family for which the index bundle arises simply from the smooth family of null spaces. Thus we have

Proposition 6. *The (virtual) index bundle of $\eth^+$ with generalized Atiyah-Patodi-Singer boundary condition corresponding to a spectral section P for $\eth^0$ and the index bundle of $\widetilde{\eth}^+ = \eth^+ + A_P^+$ with A_P^+ given by (8.7) represent the same class in $K^0(B)$.*

Proof. As τ varies the virtual index bundles correspond to a smooth homotopy of locally Fredholm problems, so the K-class of the bundle remains unchanged. q.e.d.

9. Bismut superconnection

The idea of using a superconnection to derive a local form of the family index theorem is due to Quillen [22]. Bismut in [8] carried out the program and it was extended to the family version of the Atiyah-Patodi-Singer index theorem by Bismut and Cheeger [9]. Here we construct the Bismut superconnection in the setting of b-geometry relying heavily on the treatment of the boundaryless case by Berline, Getzler and Vergne in [13, Chapter 10].

Consider a family of exact b-manifolds as in Section 1. Suppose that the fibration is given a connection, which is to say the choice of a splitting:

$$ {}^bTM = T_HM \oplus {}^bT(M/B), \tag{9.1} $$

so projection gives an identification $T_HM \cong \phi^*TB$. Let π_H and π_V be the projections onto the horizontal and vertical parts, i.e., the first and second summands.

This decomposition and the fibre metric together yield a degenerate inner product on ${}^bTM$, vanishing identically on T_HM :

$$ \langle X, Y\rangle_0 = \langle \pi_V X, \pi_V Y\rangle. \tag{9.2} $$

Similarly the dual splitting

$$ {}^bT^*M = T_H^*M \oplus {}^bT^*(M/B). \tag{9.3} $$

and the dual fibre metric define a degenerate inner product on ${}^bT^*M$:

$$ \langle \omega, \eta\rangle_0 = \langle \pi_V\omega, \pi_V\eta\rangle. \tag{9.4} $$

The Levi-Civita connection on the fibres extends to a full connection on ${}^bT(M/B)$, $\nabla^{M/B}$, such that the metric is covariant-constant; it is determined by

$$ \begin{aligned} 2\langle \nabla_X^{M/B} Y, Z\rangle_0 =& \langle [X,Y], Z\rangle_0 + \langle [Z,X], Y\rangle_0 \\ & - \langle [Y,Z], X\rangle_0 + X\langle Y, Z\rangle_0 \\ & + Y\langle Z, X\rangle_0 - Z\langle X, Y\rangle_0 \end{aligned} \tag{9.5} $$

for each $X \in \mathcal{V}_b(M),\ Y, Z \in \mathcal{C}^\infty(M; {}^bT(M/B))$.

The two basic tensors associated to the connection and fibre metric are the second fundamental form and the curvature. As 3-tensors on M they are defined by

$$ \begin{aligned} 2S(X,Y)(Z) =& 2\langle \nabla^{M/B}_{\pi_H Z} \pi_V X - [\pi_H Z, \pi_V X], Y\rangle_0 \\ =& (\pi_H Z)\langle X, Y\rangle_0 - \langle [\pi_H Z, \pi_V X], Y\rangle_0 \\ & - \langle [\pi_H Z, \pi_V Y], X\rangle_0 \end{aligned} \tag{9.6} $$

and

$$ \Omega(X,Y)(Z) = -\langle [\pi_H X, \pi_H Y], Z\rangle_0. \tag{9.7} $$

It follows easily from the definitions that

$$\begin{aligned} S \in & \mathcal{C}^\infty(M; {}^bT^*(M/B) \otimes {}^bT^*(M/B) \otimes \phi^*TB), \\ \Omega \in & \mathcal{C}^\infty(M; \phi^*T^*B \otimes \phi^*T^*B \otimes {}^bT^*(M/B)) \end{aligned} \tag{9.8}$$

are respectively even and odd in the first two arguments. The mean curvature $k \in \mathcal{C}^\infty(M; \phi^*T^*B)$ is the trace of S in the first two variables, with respect to the fibre metric. The 3-cotensor

$$\begin{aligned} \omega(X)(Y,Z) = & S(X,Z)(Y) - S(X,Y)(Z) \\ & + \frac{1}{2}\Omega(X,Z)(Y) - \frac{1}{2}\Omega(X,Y)(Z) \\ & + \frac{1}{2}\Omega(Y,Z)(X) \end{aligned} \tag{9.9}$$

is particularly important because it occurs below in the limiting Levi-Civita connection.

Next we look at the boundary behaviour of these tensors. Since the family of metrics $g_{M/B}$ is exact we have

$$\nabla^{M/B}_{x\frac{\partial}{\partial x}} \equiv 0 \quad \text{at } \partial M. \tag{9.10}$$

There is a naturally induced connection on $T(\partial M/B)$, $\nabla^{\partial M/B}$, which coincides with the analogue of (9.5) for the family of Riemannian manifolds, $\partial\phi : \partial M \longrightarrow B, g_{\partial M/B}$, with connection

$$T(\partial M) = (T_H M)_{\partial M} \oplus T(\partial M/B).$$

We denote by $\Omega^{\partial M}$ and $k^{\partial M}$ the curvature tensor and the mean curvature associated to this fibration. Then (9.10) implies that

$$\begin{aligned} k|_{\partial M} = & k^{\partial M}, \\ \Omega(X,Y)(x\frac{\partial}{\partial x}) \equiv & 0 \quad \text{at } \partial M, \\ \Omega(X,Y)(Z) = & \Omega^{\partial M}(X,Y)(Z) \quad \text{at } \partial M, \\ & \forall\ X, Y \in \mathcal{C}^\infty(M; T_H M), \\ & Z \in \mathcal{C}^\infty(\partial M; T(\partial M/B)). \end{aligned} \tag{9.11}$$

If ∇^B is the Levi-Civita connection of some Riemann metric g_B on B, a connection can be defined on ${}^bT^*M$ using the decomposition (9.1):

$$\nabla^\oplus = \phi^*\nabla^B \oplus \nabla^{M/B}. \tag{9.12}$$

The degenerate inner products (9.4) is simply the limit as u tends to zero of the family of b-metrics $g^u = ug^B \oplus g^{M/B}$. Let ∇^u be the Levi-Civita connection on ${}^bT^*M$ associated to g^u. The limiting Levi-Civita connection, $\nabla^0 = \lim\limits_{u\to 0} \nabla^u$, (see [13, §10.3]) is related to $\nabla^{\oplus}$ by:

$$\nabla^0 = \nabla^{\oplus} + \frac{1}{2}\tau(\omega), \tag{9.13}$$

where $\tau : {}^b\Lambda^2 M \longrightarrow \hom({}^bT^*M)$ is defined by

$$\frac{1}{2}\tau(\xi\wedge\eta)\mu = \langle \xi, \mu\rangle_0 \eta - \langle \eta, \mu\rangle_0 \xi, \tag{9.14}$$

so $\tau(\omega)$ is a 1-form with values in $\hom({}^bT^*M)$. Let $\mathrm{Cl}_0(M)$ be the Clifford algebra defined by the degenerate inner product $\langle , \rangle_0$, the quotient of the tensor algebra of ${}^bT^*M$ by the relations

$$\xi\cdot\zeta + \zeta\cdot\xi = 2\langle \xi, \zeta\rangle_0. \tag{9.15}$$

Clearly $\mathrm{Cl}_0(M)$ is the graded tensor product

$$\mathrm{Cl}_0(M) = (\phi^*\Lambda^* B) \otimes \mathrm{Cl}_\phi(M) \tag{9.16}$$

of the lift of the exterior algebra of B and the non-degenerate Clifford algebra $\mathrm{Cl}_\phi(M)$. The connection ∇^0 in (9.9) preserves the relations (9.15) and so induces a connection on $\mathrm{Cl}_0(M)$. The original fibre connection on E can be extended to a full connection which is still unitary and consistent with the Clifford action by $\mathrm{Cl}_\phi(M)$. It suffices to do this locally and then use a partition of unity. Since the fibres are even-dimensional locally the bundle decomposes as a tensor product $S \otimes G$ where S is the spinor bundle of a local spin structure, with its Hermitian Clifford action, and on G the Clifford action is trivial. The fibre connection on S is necessarily that induced by the Levi-Civita connection on ${}^bT(M/B)$; the extension of the fibre Levi-Civita connection on ${}^bT(M/B)$ to ∇^0 therefore induces an Hermitian Clifford connection on S. Taking any Hermitian extension of the connection on G yields a local unitary Clifford connection on E. By a partition of unity this gives a choice of connection on E which will be denoted ∇^E.

Assuming that E is a $\mathrm{Cl}_\phi(M)$ module the superbundle

$$\mathbb{E} = \phi^*\Lambda^* B \otimes E, \tag{9.17}$$

where we are again considering the graded tensor product, has a natural structure as a $\mathrm{Cl}_0(M)$ module coming from the tensor product decomposition (9.16); a differential form α on the first factor of (9.16) acts by exterior multiplication on the first factor of (9.17) whereas a vertical Clifford element acts on the second factor of (9.17) according to the $\mathrm{Cl}_\phi(M)$-module structure of E. We denote this Clifford action of $\mathrm{Cl}_0(M)$ on $\mathbb{E}$ by m_0.

Definition 3. A b-superconnection on E is a b-differential operator

$$\mathbb{A} : \mathcal{C}^\infty(M;\mathbb{E}) \longrightarrow \mathcal{C}^\infty(M;\mathbb{E}) \tag{9.18}$$

of odd parity such that

$$\begin{gathered}\mathbb{A}\left((\phi^*\alpha)u\right) = \left(\phi^*(d\alpha)\right)u + (-1)^k(\phi^*\alpha)\mathbb{A}u \\ \forall\ \alpha \in \mathcal{C}^\infty(B;\Lambda^k),\ \ u \in \mathcal{C}^\infty(M;\mathbb{E}).\end{gathered} \tag{9.19}$$

From (9.19) it follows that $\mathbb{A}$ is determined by its action on $\mathcal{C}^\infty(M;E)$, which decomposes into

$$\mathbb{A} = \sum_{j=0}^{\dim B} \mathbb{A}_{[j]},\ \ \mathbb{A}_{[j]} : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;\phi^*\Lambda^j(B)\otimes E). \tag{9.20}$$

It follows that $\mathbb{A}_{[1]}$ is a B-connection on E, that is,

$$\mathbb{A}_{[1]} : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;\phi^*\Lambda^1(B)\otimes E)$$

satisfies (9.19) but with $\alpha \in \mathcal{C}^\infty(B)$, $u \in \mathcal{C}^\infty(M;E)$. Initially we shall also demand that $\mathbb{A} \in \mathrm{Diff}^1_b(M;\mathbb{E})$ from which it also follows that

$$\mathbb{A}_{[j]} \in \mathrm{Diff}^1_{b,\phi}(M;E,\phi^*\Lambda^j(B)\otimes E) \quad \text{for } j \neq 1. \tag{9.21}$$

The superconnection is said to extend the B-connection $\mathbb{A}_{[1]}$ and to be adapted to the operator $\mathbb{A}_{[0]}$.

Bismut's superconnection is a (degenerate) Dirac operator on the bundle $\mathbb{E}$ associated to a connection which is Clifford with respect to the $\mathrm{Cl}_0(M)$-module structure on $\mathbb{E}$ and the limiting Levi-Civita connection ∇^0 on $\mathrm{Cl}_0(M)$. The connection on $\mathbb{E}$ is:

$$\nabla^{\mathbb{E},0} = \phi^*(\nabla^B)\otimes \mathrm{Id} + \mathrm{Id}\otimes\nabla^E + \frac{1}{2}m_0(\omega). \tag{9.22}$$

Here ∇^B is the Levi-Civita connection on Λ^*B, and ∇^E is the above unitary connection on E which is Clifford for the Cl_ϕ-action. The tensorial term $m_0(\omega)$ is the 1-form defined in (9.9) with values in ${}^b\Lambda^2(M)$ acting through Clifford multiplication on $\mathbb{E}$.

Definition 4. The Bismut superconnection $\mathbb{A}$ is defined as the Dirac operator on $\mathbb{E}$ associated to the $\mathrm{Cl}_0(M)$ action m_0 and the Clifford connection (9.22).

The Bismut superconnection is independent of the choice of horizontal metric g_B used in its definition. There is an explicit formula for $\mathbb{A}$ acting on $\mathcal{C}^\infty(M;E)$:

$$\mathbb{A} = \eth + \mathbb{A}_{[1]} + \mathbb{A}_{[2]} \tag{9.23}$$

with

$$\begin{aligned} \mathbb{A}_{[1]} &= -\, i \sum_\alpha \epsilon(f^\alpha)(\nabla^E_{f_\alpha} + \frac{1}{2}k(f_\alpha)), \\ \mathbb{A}_{[2]} &= -\, \frac{1}{4i} \sum_{\alpha<\beta} \sum_j \epsilon(f^\alpha)\epsilon(f^\beta)\,\mathrm{cl}(e^j)(\Omega(f_\alpha, f_\beta)(e_j)). \end{aligned} \tag{9.24}$$

In these formulae $\{f_\alpha\}$, $\{e_j\}$ are orthonormal frames in TB, ${}^bT(M/B)$ with dual frames $\{f^\alpha\}$, $\{e^j\}$; $\epsilon(f^\alpha)$ denotes as usual exterior multiplication by the one-form f^α.

The main feature of the Bismut superconnection is that its curvature $\mathbb{A}^2 \in \mathrm{Diff}^2_{b,\phi}(M;\mathbb{E})$ can be computed via a Lichnerowicz formula. For each $z \in B$ let $\Delta^{M/B}_z$ be the Laplacian on M_z associated to the connection $\nabla^{\mathbb{E},0}|_{M_z}$, and let $\Delta^{M/B} \in \mathrm{Diff}^2_{b,\phi}(M;E)$ be the corresponding family of operators. Let $R^{M/B}$ be the curvature of the connection $\nabla^{M/B}$; we denote by $S_{M/B}$ the associated scalar curvature. Finally let $\hom_{\mathrm{Cl}_\phi}(E)$ be the bundle of endomorphisms of E commuting with the vertical Clifford action, and let $K'_E \in \mathcal{C}^\infty(M; {}^b\Lambda^2 M \otimes \hom_{\mathrm{Cl}_\phi}(E))$ be the twisting curvature of the bundle E with respect to the connection ∇^E (*cf.* [13], [20]). For the square of the Bismut superconnection the following extension of Lichnerowicz' formula holds

$$\begin{aligned} \mathbb{A}^2 =& \Delta^{M/B} + \frac{1}{4} S_{M/B} \\ &- \frac{1}{2} \sum_{a,b} m_0(e^a) m_0(e^b) K'_E(e_a, e_b), \end{aligned} \tag{9.25}$$

where the sum is over both vertical and horizontal tangent vectors.

We conclude this section by pointing out that if

$$A \in \Psi^{-\infty}_{b,\phi}(M;E),$$

then the perturbed Bismut superconnection

$$\widetilde{\mathbb{A}} = (\eth + A) + \mathbb{A}_{[1]} + \mathbb{A}_{[2]} \tag{9.26}$$

with $\mathbb{A}_{[1]}$, $\mathbb{A}_{[2]}$ given by (9.24) is such that its square

$$\widetilde{\mathbb{A}}^2 \in \Psi^2_{b,\phi}(M, \mathbb{E}) \tag{9.27}$$

is the sum of $\mathbb{A}^2$ and an element in $\Psi^{-\infty}_{b,\phi}(M, \mathbb{E})$.

10. Cl(1)-superconnections

In the odd-dimensional case we define a superconnection which is associated with a Cl(1)-module, following the ideas of Quillen (see also Getzler [16] for a similar development).

Let $\psi : M' \longrightarrow B$ be a fibration with odd-dimensional fibres with fibre metric, and suppose E^0 is a bundle over M which is a unitary Clifford module for the fibre Clifford algebra and carries a unitary Clifford connection. The bundle E^0 is not assumed to have a $\mathbb{Z}_2$ grading. Instead we give $E = E^0 \oplus E^0$ a Cl(1) action by choosing a smooth, odd, self-adjoint involution σ; we shall take

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{10.1}$$

Notice that if

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{10.2}$$

then

$$\sigma = iR\gamma. \tag{10.3}$$

Thus $\mathbb{C}^2$ can be identified with the algebra, Cl(1), generated by σ and Id, so $E = E^0 \otimes \mathbb{C}^2 = E^0 \otimes \mathrm{Cl}(1)$. The bundle $\mathbb{E}$ is defined by (9.17) as before. The involution σ acts on $\mathbb{E}$ as $\mathrm{Id} \otimes \sigma$.

In general a homomorphism A, of E as a Clifford module, is a homomorphism of E as a bundle which commutes with the action of σ on the right, $A(u\sigma) = (Au)\sigma$. Such a homomorphism decomposes into

$$A = A' + \sigma A'',$$

where A' and A'' are homomorphisms of E^0 lifted to act diagonally on E. This makes the space of homomorphisms into a superalgebra, with the odd terms being those of the form $\sigma A''$. The supertrace functional is defined on these (Cl(1)-module) homomorphisms by

$$\operatorname{Str}_\sigma(A) = \operatorname{tr}(A''). \tag{10.4}$$

The supertrace functional extends to be form-valued on the space of Λ^*-linear homomorphisms of $\mathbb{E}$ as before.

The definition of a superconnection on E can be applied, with the additional condition that it be a map of Cl(1)-modules; i.e., that it commutes with the right action of the involution σ. We generally denote such Cl(1)-superconnections by $\mathbb{B}$:

$$\begin{gathered}\mathbb{B} : \mathcal{C}^\infty(M';\mathbb{E}) \longrightarrow \mathcal{C}^\infty(M';\mathbb{E}),\\ \mathbb{B}\left((\phi^*(\alpha)u\right) = \left(\phi^*(d\alpha)\right)u + (-1)^k(\phi^*(\alpha)\mathbb{B}u,\\ \forall\ \alpha \in \mathcal{C}^\infty(B;\Lambda^k),\ u \in \mathcal{C}^\infty(M';\mathbb{E}),\\ \mathbb{B}(u\sigma) = (\mathbb{B}u)\,\sigma.\end{gathered} \tag{10.5}$$

We still insist that $\mathbb{B}$ be odd.

To see how Cl(1)-superconnections arise naturally in our problem consider the Bismut b-superconnection (9.23) for the original fibration with boundary $\phi : M \longrightarrow B$. Let $M' = \partial M$ and consider the boundary fibration $\partial\phi : M' \longrightarrow B$. The restriction of ∇^E to the boundary induces through the identification (1.13) a connection on the bundle $E^0 \oplus E^0$ over M' :

$$\nabla^{E^0\oplus E^0} = M \cdot \nabla^E|_{\partial M} \cdot M^{-1}.$$

Let us consider the B-connection on $E^0 \oplus E^0$:

$$\mathbb{B}_{[1]} = -i\sum_\alpha \epsilon(f^\alpha)(\nabla^{E^0\oplus E^0}_{f_\alpha} + \frac{1}{2}k^{\partial M}), \tag{10.6}$$

and let $\mathbb{B}_{[2]} \in \mathcal{C}^\infty(M';\hom((\partial\phi)^*\Lambda^*B \otimes E^0))$ be given by

$$\mathbb{B}_{[2]} = -\frac{1}{4i}\sum_{\alpha<\beta}\sum_j \epsilon(f^\alpha)\epsilon(f^\beta)\,\mathrm{cl}_{\partial\phi}(g^j)(\Omega^{\partial M}(f_\alpha, f_\beta)(g_j)) \tag{10.7}$$

with $\{g_j\}$ an orthonormal frame of $T(\partial M/B)$ with respect to $g_{\partial M/B}$.

Proposition 7. *The indicial family of the Bismut b-superconnection is given by*

$$M \cdot I(\mathbb{A}, \lambda) \cdot M^{-1} = \gamma\lambda + \sigma\eth^0 + \mathbb{B}_{[1]} + \sigma\mathbb{B}_{[2]} \\ = \gamma\lambda + \mathbb{B} \tag{10.8}$$

with $\mathbb{B}$ *a* $\mathrm{Cl}(1)$*-superconnection for the boundary fibration*

$$\partial\phi : \partial M = M' \longrightarrow B.$$

Proof. It suffices to consider the individual terms in (9.24). The indicial family of $\eth$ is given by formula (1.14) whereas the indicial families of $\mathbb{A}_{[1]}$ and $\mathbb{A}_{[2]}$ are simply given by the restriction to the boundary; by the boundary behaviour of Ω and k from formula (9.11) the proposition follows. q.e.d.

11. Heat kernels

The structure of the heat kernel of the Laplacian of an exact b-metric is examined in detail in [20]. In the Appendix that discussion is extended to treat families of self-adjoint operators. Here it is further refined to deal with the somewhat more general case which arises for the operator $\mathbb{A}^2$. Although we only consider the Bismut superconnection, there is no problem in generalizing the discussion.

Our goal is to show the existence of an element

$$e^{-t\mathbb{A}^2} \in \mathcal{C}^\infty(\mathbb{R}^+; \Psi^{-\infty}_{b,\phi}(M; \mathbb{E})),$$

which, for each $z \in B$, gives the unique solution to the heat equation along the fibre M_z :

$$\left(\frac{d}{dt} + \mathbb{A}_z^2\right) H_z = 0 \\ \lim_{t\downarrow 0} H_z v = v \quad \forall\ v \in \mathcal{C}^\infty(M_z; E) \otimes \Lambda_z^* B. \tag{11.1}$$

Formula (9.25) shows that $\mathbb{A}^2$ is the sum of a family of formally self-adjoint elliptic second order b-differential operators, $P \in \mathrm{Diff}^2_{b,\phi}(M, E)$, with positive principal symbol acting as a multiple of the identity on forms on the base and an endomorphism, P^+, of E with coefficients differential forms on the base of degree greater than 0. In the Appendix it is shown that the heat kernels of the $P_z \in \mathrm{Diff}^2_b(M_z; E)$ form a family

$$\exp(-tP) \in \Psi^{-2}_{\eta,\phi}(M; E).$$

Since P^+ is nilpotent, we can obtain the heat kernel of $\mathbb{A}_z^2$ from the expansion as the operator

$$e^{-t\mathbb{A}_z^2} = e^{-tP_z} + \sum_{k>0} (-t)^k I_k. \tag{11.2}$$

Here

$$I_k = \int_{\Delta^k} e^{-\sigma_0 t P_z} P_z^+ e^{-\sigma_1 t P_z} P_z^+ \cdots e^{-\sigma_k t P_z} d\sigma \tag{11.3}$$

with $\Delta^k = \{(t_1, \dots, t_k); 0 \le t_1 \le \cdots \le t_k \le 1\}$ parametrized by $\sigma_0 = t_1, \dots, \sigma_i = t_{i+1} - t_i, \dots, \sigma_k = 1 - t_k$. The nilpotence of P^+ means that the sum in (11.2) is finite. Moreover from the properties of the heat space described in the appendix each element is clearly in

$$\Psi_{\eta,\phi}^{-2}(M_z; E) \otimes \Lambda_z^* B,$$

since $P_z^+ \in \mathcal{C}^\infty(M_z; \hom(E)) \otimes \Lambda_z^* B$. It is straightforward to show that (11.2) is a solution of the heat equation (11.1) so it follows that

$$e^{-t\mathbb{A}^2} \in \Psi_{\eta,\phi}^{-2}(M; \mathbb{E}). \tag{11.4}$$

We need to further extend this result to cover the case of the perturbed Bismut superconnection

$$\widetilde{\mathbb{A}} = (\eth + A) + \mathbb{A}_{[1]} + \mathbb{A}_{[2]} \tag{11.5}$$

with $A \in \Psi_{b,\phi}^{-\infty}(M; E)$. In terms of the definition, (A.23), of the enlarged fibre heat calculus, $\Psi_{\eta,\phi}^{p,q}(M; \mathbb{E})$, we find

Proposition 8. *For any $A \in \Psi_{b,\phi}^{-\infty}(M; E)$ there is a unique solution*

$$H(t) = e^{-t\widetilde{\mathbb{A}}^2} \in \Psi_{\eta,\phi}^{-2,1}(M; \mathbb{E})$$

to the heat equation

$$\begin{aligned} \left(\frac{d}{dt} + \widetilde{\mathbb{A}}^2\right) H &= 0 \\ \lim_{t \downarrow 0} H(t) &= \mathrm{Id}\,. \end{aligned} \tag{11.6}$$

Proof. As in the discussion above, leading to (11.4), the main step is to analyze the heat kernel of $Q = P + A$ with $A \in \Psi^{-\infty}_{b,\phi}(M;E)$. Let $H^{(0)} = \exp(-tP) \in \Psi^{-2}_{\eta,\phi}(M;E)$. The composition formula in the Appendix, Proposition 20, shows that

$$\begin{aligned}\left(\frac{d}{dt}+Q\right)H^{(0)} =& R^{(0)}(t),\\ R^{(0)} =& Ae^{-tP} \in \mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)).\end{aligned} \tag{11.7}$$

We proceed to remove the error term by modifying $H^{(0)}$.

The first step is to remove the Taylor series of $R^{(0)}$ at $\mathrm{bf}(M^2_{\eta,\phi})$ (see the Appendix for the notation). We initially look for

$$H^{(0)}_1 \in t\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E))$$

such that

$$\left(\frac{d}{dt}+Q\right)H^{(0)}_1 = R^{(0)}(t) + R^{(0)}_1(t) \tag{11.8}$$

with $R^{(0)}_1 \in \mathcal{C}^\infty([0,\infty),\rho_{bf}\Psi^{-\infty}_{b,\phi}(M;E))$. This means that a solution to the indicial equation

$$\left(\frac{d}{dt}+I(Q)\right)G = I(R^{(0)}) \tag{11.9}$$

must be found in $t\mathcal{C}^\infty([0,\infty),\Psi^{-\infty}_{b,\phi}(\overline{N^+\partial M};E)$. The solution will be $\mathbb{R}^+$-invariant along the normal fibres. Taking the Mellin transform, (11.9) and the requirement that the solution vanish at $t=0$ amounts to a family of inhomogeneous Cauchy problems, depending parametrically on λ, on the closed fibres of ∂M. Composition with the heat kernel of the indicial family $\exp(-tI(Q,\lambda))$ then provides the required solution as in [20]. This gives $H^{(0)}_1$ satisfying (11.8) and thus the element $H^{(0)} - H^{(0)}_1 \in \Psi^{-2,1}_{\eta,\phi}(M;E)$ such that

$$\left(\frac{d}{dt}+Q\right)(H^{(0)} - H^{(0)}_1) \in \mathcal{C}^\infty([0,\infty),\rho_{bf}\Psi^{-\infty}_{b,\phi}(M;E)). \tag{11.10}$$

Proceeding inductively in the usual fashion we can reconstruct the whole Taylor series of $R^{(0)}$ at $\mathrm{bf}(M^2_{\eta,\phi})$, obtaining an element

$$H^{(1)} \in \mathcal{C}^\infty\big([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)\big)$$

such that for $H = H^{(0)} - H^{(1)}$

$$\left(\frac{d}{dt} + Q\right) H = R \in \mathcal{C}^\infty([0,\infty), \Psi_\phi^{-\infty,\infty}(M;E)). \tag{11.11}$$

Since this error term is a family of Volterra operators with kernel vanishing rapidly at the boundary, a parametrix H for the heat equation (11.6) has been found.

Following the standard procedure we can obtain the heat kernel by iteration:

$$e^{-tQ} = H + \sum_{k>1} \int_{t\Delta^k} H(t-t_k)R(t_k - t_{k-1}) \cdots R(t_1)dt \tag{11.12}$$

with $t\Delta^k = \{(t_1, \ldots, t_k); 0 \le t_1 \le \cdots \le t_k \le t\}$. Each element in the series belongs to the residual calculus $t\mathcal{C}^\infty([0,\infty), \Psi_\phi^{-\infty,\infty}(M;E))$, and the usual Volterra estimates show that the series converges in each C^l-norm, $l \ge 0$, to an element in the same space.

The same argument applies to $\widetilde{\mathbb{A}}^2 = \mathbb{A}^2 + \widetilde{A}$ with $\widetilde{A} \in \Psi_{b,\phi}^{-\infty}(M;\mathbb{E})$. Thus we have constructed the heat kernel associated to the square of the perturbed Bismut superconnection and

$$e^{-t\widetilde{\mathbb{A}}^2} - e^{-t\mathbb{A}^2} \in t\mathcal{C}^\infty([0,\infty); \Psi_{b,\phi}^{-\infty}(M;\mathbb{E})) \tag{11.13}$$

q.e.d.

12. Supertrace defect

Let E be a superbundle on a manifold with boundary X. The operators in $\Psi_b^{-\infty}(X;E)$ are not trace class on $L_b^2(X;E)$ (they are not even compact). Notice however that the restriction of their Schwartz kernel to the lifted diagonal Δ_b in X_b^2 is in a natural way an element in $\mathcal{C}^\infty(X; \hom(E) \otimes {}^b\Omega)$ under the obvious identifications

$$X \longleftrightarrow \Delta \longleftrightarrow \Delta_b.$$

Following [20] we can thus extend the (super)trace functional to these operators by defining

$$b\text{-STr}(A) = {}^{\nu}\!\!\int_X \mathrm{str}_E(A\big|_{\Delta_b})$$

(12.1)

$$ {}^{\nu}\!\!\int_X \mathrm{str}_E(A\big|_{\Delta_b}) \overset{\text{def}}{=} \lim_{\epsilon\to 0}\left(\int_{x>\epsilon} \mathrm{str}_E(A\big|_{\Delta_b}) + \log\epsilon\cdot\int_{\partial X}\mathrm{str}_E(A\big|_{\partial\Delta_b})\right)$$

for each Schwartz kernel $A \in \Psi_b^{-\infty}(X;E)$. Here $\nu \in \mathcal{C}^\infty(\partial X, N^+\partial X)$ is a fixed trivialization of the positive normal bundle to the boundary, and $x \in \mathcal{C}^\infty(X)$ a boundary defining function with $dx(\nu) = 1$ at ∂X.

If E is a superbundle on a $\mathcal{C}^\infty$ fibration $\phi : M \longrightarrow B$, then (12.1) gives a map

$$b\text{-STr} : \Psi_{b,\phi}^{-\infty}(M;E) \longrightarrow \mathcal{C}^\infty(B), \qquad b\text{-STr}(A)(z) = b\text{-STr}(A_z). \tag{12.2}$$

As in [22] we can extend (12.2) to any element $\eta\otimes A \equiv \eta A \in \Psi_{b,\phi}^{-\infty}(M;\mathbb{E})$ by

$$b\text{-STr}(\eta A)(z) = \eta(z)(\,b\text{-STr}(A_z)). \tag{12.3}$$

Since locally in the base the general element of $\Psi_{b,\phi}^{-\infty}(M;\mathbb{E})$ is a finite sum of such terms, (12.3) fixes a map

$$b\text{-STr} : \Psi_{b,\phi}^{-\infty}(M;\mathbb{E}) \longrightarrow \mathcal{C}^\infty(B, \Lambda^* B). \tag{12.4}$$

The b-supertrace is not a supertrace in the strict sense, i.e., does not always vanish on supercommutators. The following defect formula is a straightforward consequence of the corresponding identity for the b-trace of a commutator of b-pseudodifferential operators in [20]; *cf.* also [16, Corollary 5.5].

Proposition 9. *If E is a superbundle over a $\mathcal{C}^\infty$ fibration $\phi : M \longrightarrow B$, with compact manifolds with boundary as fibres and $\mathbb{E} = \phi^*\Lambda^*B \otimes E$, then for any $A \in \Psi_{b,\phi}^{\infty}(M;\mathbb{E})$ and $B \in \Psi_{b,\phi}^{-\infty}(M;\mathbb{E})$*

$$b\text{-STr}[A,B] = \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STr}\left(\partial_\lambda \mathrm{I}(\mathrm{A},\lambda)\cdot \mathrm{I}(\mathrm{B},\lambda)\right)\mathrm{d}\lambda. \tag{12.5}$$

Proof. It suffices to work locally in the base, so the operators can be taken to be of the forms $\eta A'$ and $\xi B'$ where η and ξ are $\mathcal{C}^\infty$ forms of degrees k and l, $A' \in \Psi^{\infty}_{b,\phi}(M;E)$ and $B' \in \Psi^{-\infty}_{b,\phi}(M;E)$. Then

$$[A,B] = (-1)^{|A'|l}\eta\wedge\xi[A',B'], \tag{12.6}$$

the exterior algebra being supercommutative. Thus it suffices to suppose that

$$A \in \Psi^{\infty}_{b,\phi}(M;E) \text{ and } B \in \Psi^{-\infty}_{b,\phi}(M;E).$$

If A and B have opposite parities, then $[A,B]$ and $\partial_\lambda I(A,\lambda)\cdot I(B,\lambda)$ are both odd, so both sides of (12.5) vanish. Thus we can take both A and B to be either even or odd. If they are both even, i.e.,

$$A = \begin{bmatrix} A^{++} & 0 \\ 0 & A^{--}\end{bmatrix},\ B = \begin{bmatrix} B^{++} & 0 \\ 0 & B^{--}\end{bmatrix},$$

then

$$b\text{-STr}[A,B] = b\text{-Tr}[A^{++},B^{++}] - b\text{-Tr}[A^{--},B^{--}].$$

Applying [20] this reduces to (12.5). Similarly if A and B are both odd

$$A = \begin{bmatrix} 0 & A^{+-} \\ A^{-+} & 0\end{bmatrix},\ B = \begin{bmatrix} 0 & B^{+-} \\ B^{-+} & 0\end{bmatrix},$$

then

$$b\text{-STr}[A,B] = b\text{-Tr}[A^{+-},B^{-+}] - b\text{-Tr}[A^{-+},B^{+-}].$$

Again this reduces to the right side of (12.5) on applying [20]. q.e.d.

Integration by parts in (12.5) allows it to be written

$$b\text{-STr}[A,B] = -\frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STr}\left(\mathrm{I}(\mathrm{A},\lambda)\cdot\partial_\lambda \mathrm{I}(\mathrm{B},\lambda)\right)d\lambda.$$

Notice in particular that if either operator has indicial family independent of λ, i.e., a bundle map, then the b-supertrace of the commutator vanishes.

13. b-Chern character

The Chern character of a superconnection is the trace of its heat kernel, which includes the operator trace on the fibres in the case of a fibration. To define the Chern character of a b-superconnection we replace the fibre supertrace by the b-supertrace:

$$b\text{-Ch}(\mathbb{A}) = b\text{-STr}(e^{-\mathbb{A}^2}). \tag{13.1}$$

From the properties of the heat kernels discussed in Section 11 it follows that $b\text{-Ch}(\mathbb{A})$ is a smooth form on B. Since $\mathbb{A}^2$ is even with respect to the total $\mathbb{Z}_2$-grading, and the supertrace vanishes on odd elements of $\hom(E)$, $b\text{-Ch}(\mathbb{A})$ has only terms in even degrees.

For a $\mathrm{Cl}(1)$-superconnection, where we assume that the leaves of the fibration $\psi : M' \longrightarrow B$ have no boundary, we similarly define

$$\mathrm{Ch}_{\mathrm{Cl}(1)}(\mathbb{B}) = \mathrm{STr}_{\mathrm{Cl}(1)}(e^{-\mathbb{B}^2}). \tag{13.2}$$

In this case $\mathrm{Ch}_{\mathrm{Cl}(1)}(\mathbb{B})$ has only terms of odd degree since the $\mathrm{Cl}(1)$-supertrace vanishes on even elements of $\hom(E)$, and $\mathbb{B}^2$ is even with respect to the total grading of $\mathbb{E}$.

As a first application of the defect formula (12.5) we compute the exterior derivative of the b-Chern character.

Proposition 10. *For a b-superconnection adapted to the fibre Dirac operator of a graded Clifford module over a family of exact b-manifolds,*

$$d_B\ b\text{-Ch}(\mathbb{A}) = \frac{1}{2\sqrt{\pi}}\,\mathrm{Ch}_{\mathrm{Cl}(1)}(\mathbb{B}), \tag{13.3}$$

where $\mathbb{B}$ is the induced $\mathrm{Cl}(1)$-superconnection over the boundary.

Proof. Since $\mathbb{A}_{[1]}$ is a B-connection on E, it is given in any local trivialization by $d + \gamma$ where γ is a bundle map. Using the remark following Proposition 9 we have

$$d_B\ b\text{-Ch}(\mathbb{A}) = b\text{-STr}[\mathbb{A}_{[1]}, e^{-\mathbb{A}^2}]. \tag{13.4}$$

Since $\mathbb{A}_{[j]}$, for $j \geq 2$, are by assumption bundle maps and $[\mathbb{A}, e^{-\mathbb{A}^2}] = 0$,

$$d_B\ b\text{-Ch}(\mathbb{A}) = -\ b\text{-STr}[\mathbb{A}_{[0]}, e^{-\mathbb{A}^2}]. \tag{13.5}$$

Applying (12.5) gives

$$d_B\ b\text{-Ch}(\mathbb{A}) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{STr}\left(\partial_\lambda \mathrm{I}(\mathbb{A}_{[0]}, \lambda)\mathrm{I}(\mathrm{e}^{-\mathbb{A}^2}, \lambda)\right) \mathrm{d}\lambda. \tag{13.6}$$

With $\mathrm{cl}(idx/x)$ used to identify E with $E^0 \otimes \mathbb{C}^2$ over the boundary, the indicial family of the curvature operator becomes $\mathbb{B}^2 + \lambda^2\,\mathrm{Id}$ and

$\partial_\lambda I(\mathbb{A}_{[0]}, \lambda) = \gamma$ where γ is the matrix in (10.2). The λ-integral in (13.6) can therefore be carried out, and hence we obtain (13.3), since $iR\gamma = \sigma$ (see (10.2) and (10.3)) and $\mathrm{STr}(\mathrm{F}) = \mathrm{Tr}(\mathrm{RF})$. q.e.d.

One of the basic properties of the heat kernel for the square of a generalized Dirac operator on a closed manifold is that it interpolates between the kernel of the projection onto the null space of $\eth$ at $t = \infty$, with supertrace equal to the index of $\eth^+$, and a kernel whose supertrace at $t = 0$ is computable. Since the supertrace of the heat kernel is constant in t, the index theorem follows from simply equating these two terms. In the case of a manifold with boundary with an exact b-metric the contributions of the b-supertraces from $t = \infty$ and $t = 0$ remain the same but their difference, i.e., the integral from 0 to infinity of the t-derivative of the b-supertrace of the heat kernel, is not zero but rather is the eta invariant.

For a family of Dirac operators this fundamental interpolating property is captured by the behaviour in t of the supertrace of the heat kernel $\exp(-\mathbb{A}_t^2)$ of the rescaled Bismut superconnection

$$\mathbb{A}_t = t^{\frac{1}{2}}\eth + \mathbb{A}_{[1]} + t^{-\frac{1}{2}}\mathbb{A}_{[2]}. \tag{13.7}$$

This is true in both the closed case [13] and in the case of fibrations by b-manifolds as in Section 1. We will deal with such a question in Section 15. This property granted, it is clear that the boundary contribution in the family index theorem is connected to the t-derivative of the b-supertrace of the heat kernel $\exp(-\mathbb{A}_t^2)$. The defect formula (12.5) can be applied to compute this derivative giving the basic identity underlying the proof of the index formula.

Proposition 11. *The b-Chern character of the rescaled Bismut superconnection for a Clifford module over a family of exact b-manifolds satisfies*

$$\frac{d}{dt}\ b\text{-}\mathrm{Ch}(\mathbb{A}_t) = -d_B\ b\text{-}\mathrm{STr}\left(\frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}\right) - \frac{1}{2}\widehat{\eta}(t), \tag{13.8}$$

where the eta forms are defined by

$$\widehat{\eta}(t) = \frac{1}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right). \tag{13.9}$$

Proof. By Duhamel's formula, the derivative of the heat kernel is

$$\frac{d}{dt}e^{-\mathbb{A}_t^2} = -\int_0^1 e^{-s\mathbb{A}_t^2} \cdot \frac{d\mathbb{A}_t^2}{dt} \cdot e^{-(1-s)\mathbb{A}_t^2}\, ds. \tag{13.10}$$

The indicial family of $\mathbb{A}_t^2$ is even in λ, so applying (12.5) to commute the leading term, from left to right, no boundary terms arise. Thus

$$\frac{d}{dt}\ b\text{-}\mathrm{Ch}(\mathbb{A}_t) = -\ b\text{-}\mathrm{STr}\left(\frac{d\mathbb{A}_t^2}{dt}e^{-\mathbb{A}_t^2}\right). \tag{13.11}$$

Since $\mathbb{A}_t$ is odd and commutes with $\exp(-\mathbb{A}_t^2)$, this can be written as a supercommutator:

$$\frac{d}{dt}\ b\text{-}\mathrm{Ch}(\mathbb{A}_t) = -\ b\text{-}\mathrm{STr}[\mathbb{A}_t, \frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}]. \tag{13.12}$$

Now, $d\mathbb{A}_t/dt$ is a fibre operator so (12.5) can be applied to give

$$\begin{aligned} &b\text{-}\mathrm{STr}[\mathbb{A}_t - \mathbb{A}_{[1]}, \frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}] \\ &= \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STr}\left(\partial_\lambda \mathrm{I}(\mathbb{A}_\mathrm{t} - \mathbb{A}_{[1]}, \lambda)\cdot \mathrm{I}(\frac{\mathrm{d}\mathbb{A}_\mathrm{t}}{\mathrm{dt}}, \lambda)\cdot \exp(-\mathbb{B}_\mathrm{t}^2 - \mathrm{t}\lambda^2)\right). \end{aligned} \tag{13.13}$$

Since $\partial_\lambda I(\mathbb{A}_t - \mathbb{A}_{[1]}, \lambda)$ is $t^{\frac{1}{2}}\gamma$, where γ is the matrix in (10.2), and the even part, in λ, of the middle term is $d\mathbb{B}_t/dt$, the λ-integral in (13.13) becomes

$$\frac{1}{2\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right). \tag{13.14}$$

Thus (13.13) is reduced to the eta term in (13.8), given by (13.9). The remaining term is

$$-\ b\text{-}\mathrm{STr}[\mathbb{A}_{[1]}, \frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}] = -d_B\ b\text{-}\mathrm{STr}\left(\frac{d\mathbb{A}_t}{dt}e^{-\mathbb{A}_t^2}\right).$$

q.e.d.

In order to prove the index formula we must extend this result to the perturbed Bismut superconnection

$$\mathbb{A}_t(\epsilon) = t^{\frac{1}{2}}\left(\eth + \chi(t)A_P(\epsilon)\right) + \mathbb{A}_{[1]} + t^{-\frac{1}{2}}\mathbb{A}_{[2]}. \tag{13.15}$$

Here $A_P(\epsilon)$ is the $\mathbb{Z}_2$-graded operator

$$A_P(\epsilon) = \begin{pmatrix} 0 & A_P^- \\ A_P^+ & 0 \end{pmatrix}$$

constructed from the operator A_P^+ of Lemma 9 with $A_P^- = (A_P^+)^*$. It follows that 0 is never the imaginary part of an indicial root of $\eth + A_P(\epsilon)$. The cutoff function $\chi \in \mathcal{C}^\infty(\mathbb{R})$, with $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$ is inserted to ensure the regularity of the b-Chern character at $t = 0$. Using (1.13) to identify E with $E^0 \otimes \mathbb{C}^2$ over the boundary and (8.5) we have, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} I(\mathbb{A}_t(\epsilon), \lambda) = & t^{\frac{1}{2}} \left(\gamma\lambda + \sigma\eth^0 + \sigma\chi(t)\widehat{\rho}_\epsilon(\lambda) A_P^0\right) \\ & + \mathbb{B}_{[1]} + t^{-\frac{1}{2}}\sigma\mathbb{B}_{[2]}. \end{aligned} \tag{13.16}$$

Introducing the family of Cl(1)-superconnections

$$\mathbb{B}_t(\lambda, \epsilon) = t^{\frac{1}{2}}\sigma\chi(t)\widehat{\rho}_\epsilon(\lambda) A_P^0 + \mathbb{B}_t, \quad \lambda \in \mathbb{R} \tag{13.17}$$

we find

$$I(\mathbb{A}_t^2(\epsilon), \lambda) = t\lambda^2 + \mathbb{B}_t(\lambda, \epsilon)^2. \tag{13.18}$$

Since $\widehat{\rho}_\epsilon$ is even in λ, we can use the argument given in the proof of Proposition 11, namely Duhamel's formula and (12.5), to conclude that also in this case

$$\frac{d}{dt}\ b\text{-Ch}(\mathbb{A}_t(\epsilon)) = -\ b\text{-STr}[\mathbb{A}_t(\epsilon), \frac{d\mathbb{A}_t(\epsilon)}{dt} e^{-\mathbb{A}_t^2(\epsilon)}]. \tag{13.19}$$

By applying (12.5) to the right-hand side we obtain

$$\begin{aligned} \frac{d}{dt}\ b\text{-Ch}(\mathbb{A}_t(\epsilon)) = & -d_B\ b\text{-STr}\left(\frac{d\mathbb{A}_t(\epsilon)}{dt} e^{-\mathbb{A}_t^2(\epsilon)}\right) \\ & - \frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{STr}(\mathrm{t}^{\frac{1}{2}}(\gamma + \sigma\chi(\mathrm{t})\frac{\mathrm{d}\widehat{\rho}_\epsilon}{\mathrm{d}\lambda}\mathrm{A}_\mathrm{P}^0)) \\ & \times \left(\frac{d}{dt}(t^{\frac{1}{2}}(\lambda\gamma + \sigma\chi(t)\widehat{\rho}_\epsilon(\lambda) A_P^0)) + \frac{d\mathbb{B}_t}{dt}\right) \\ & \times e^{-(t\lambda^2 + \mathbb{B}_t(\lambda,\epsilon)^2)} d\lambda. \end{aligned}$$

Using again the fact that $\widehat{\rho}_\epsilon(\lambda)$ is even in λ, whereas $d\widehat{\rho}_\epsilon/d\lambda$ is odd, the second term on the right-hand side reduces to $-\widehat{\eta}_P(t,\epsilon)/2$ with

$$
\begin{aligned}
\widehat{\eta}_P(t,\epsilon) &\overset{\text{def}}{=} \frac{i}{\pi}\int\limits_{\mathbb{R}} \left(\mathrm{STr}(\mathrm{t}^{\frac{1}{2}}\gamma)(\frac{\mathrm{d}\mathbb{B}_{\mathrm{t}}(\lambda,\epsilon)}{\mathrm{dt}})\mathrm{e}^{-(\mathrm{t}\lambda^2+\mathbb{B}_{\mathrm{t}}(\lambda,\epsilon)^2)}\right) d\lambda \\
&\quad + \frac{i}{\pi}\int\limits_{\mathbb{R}} \left(\mathrm{STr}\frac{1}{2}(\sigma\chi(\mathrm{t})\frac{\mathrm{d}\widehat{\rho}_\epsilon}{\mathrm{d}\lambda}\mathrm{A}^0_{\mathrm{P}})(\gamma\lambda)\mathrm{e}^{-(\mathrm{t}\lambda^2+\mathbb{B}_{\mathrm{t}}(\lambda,\epsilon)^2)}\right) d\lambda. \qquad (13.20)
\end{aligned}
$$

Thus for the perturbed Bismut superconnection (13.15) we have :

$$
\begin{aligned}
&\frac{d}{dt}\ b\text{-}\mathrm{Ch}(\mathbb{A}_t(\epsilon)) \\
&\quad = -d_B\ b\text{-}\mathrm{STr}\left(\frac{d\mathbb{A}_t(\epsilon)}{dt}e^{-\mathbb{A}_t^2(\epsilon)}\right) \qquad (13.21)\\
&\quad\quad -\frac{1}{2}\widehat{\eta}_P(t,\epsilon).
\end{aligned}
$$

The family index theorem will be obtained by integration in t, from 0 to ∞, of formula (13.21).

14. Normalized $\widehat{\eta}$ form

In this section we analyze the eta form (13.20). First of all we must check that integration in t is permissible.

Proposition 12. *The integral*

$$
\widehat{\eta}_P(\epsilon) = \int\limits_0^\infty \widehat{\eta}_P(t,\epsilon)dt \qquad (14.1)
$$

converges in $\mathcal{C}^\infty(B,\Lambda^*(B))$.

Proof. We can write $\widehat{\eta}_P(t,\epsilon)$ as

$$
\begin{aligned}
\widehat{\eta}_P(t,\epsilon) =& \frac{1}{\pi}\int\limits_{\mathbb{R}} t^{\frac{1}{2}}\,\mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}_t(\lambda,\epsilon)}{dt}e^{-\mathbb{B}_t(\lambda,\epsilon)^2}\right)e^{-t\lambda^2}d\lambda \\
&-\frac{1}{\pi}\int\limits_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{1}{2}\sigma\chi(t)\lambda\frac{d\widehat{\rho}_\epsilon}{d\lambda}A^0_P e^{-\mathbb{B}_t(\lambda,\epsilon)^2}\right)e^{-t\lambda^2}d\lambda, \qquad (14.2)
\end{aligned}
$$

where we have used the fact that $\gamma\sigma = -\sigma\gamma$. Since $\mathrm{null}(\eth^0_z+\widehat{\rho}_\epsilon(\lambda)(A^0_P)_z) = 0$ for each $\lambda\in\mathbb{R}$ and each $z\in B$, the heat kernel $\exp(-\mathbb{B}_t(\lambda,\epsilon)^2)$

converges exponentially to 0 as $t \to \infty$, and uniformly in $\lambda \in [-K, K]$. It follows that $\widehat{\eta}_P(t,\epsilon)$ is integrable as $t \to \infty$. The large time convergence of the integral over $|\lambda| > K$ can be shown by conjugation with an exponential factor. As far as the convergence as $t \to 0$ is concerned, the insertion of $\chi(t)$ means that this is required only for the rescaled Bismut superconnection. By [10] (see also [13]) we therefore know that for t small

$$\| \operatorname{STr}_{\mathrm{Cl}(1)} \left(\frac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_t^2} \right) \|_k \le C(k) t^{-\frac{1}{2}},$$

where $\| \cdot \|_k$ denotes the C^k-norm. q.e.d.

Next consider the Cl(1)-superconnection

$$\widetilde{\mathbb{B}}_t = \mathbb{B}_t + t^{\frac{1}{2}} \sigma \chi(t) A^0_P. \tag{14.3}$$

The argument used in the proof of Proposition 12 shows that the differential form

$$\widehat{\eta}_P(t) = \frac{1}{\sqrt{\pi}} \operatorname{STr}_{\mathrm{Cl}(1)} \left(\frac{d\widetilde{\mathbb{B}}_t}{dt} e^{-\widetilde{\mathbb{B}}_t^2} \right), \tag{14.4}$$

is integrable in $[0, \infty)$.

The eta form appearing in the final index formula is

$$\widehat{\eta}_P = \int_0^\infty \widehat{\eta}_P(t) dt, \tag{14.5}$$

which will be seen below to be the limit of $\widehat{\eta}_P(\epsilon)$ as $\epsilon \downarrow 0$. In order to compare the two forms $\widehat{\eta}_P(\epsilon)$ and $\widehat{\eta}_P$ we will now compute the variation in ϵ of $\widehat{\eta}_P(t,\epsilon)$.

Proposition 13. *The differential forms defined by (14.1) are smooth in ϵ and t and*

$$\begin{aligned} \frac{d}{d\epsilon} \widehat{\eta}_P(t,\epsilon) = & \frac{d}{dt} \left(\frac{1}{\pi} \int_{\mathbb{R}} \operatorname{STr}_{\mathrm{Cl}(1)} \left(t^{\frac{1}{2}} \frac{d\mathbb{B}_t(\lambda,\epsilon)}{d\epsilon} e^{-t\lambda^2 - \mathbb{B}_t(\lambda,\epsilon)^2} \right) d\lambda \right) \\ & + d_B \alpha(t,\epsilon). \end{aligned} \tag{14.6}$$

Proof. Smoothness in ϵ and t is clear from the definition. To simplify the notation set $\mathbb{D} = \mathbb{B}_t(\lambda,\epsilon)$. Consider the vector field

$$L = t^{\frac{1}{2}} \frac{\partial}{\partial t} - \frac{\lambda t^{-\frac{1}{2}}}{2} \frac{\partial}{\partial \lambda},$$

which is divergence-free with respect to the measure $dtd\lambda$ and annihilates the function $\exp(-t\lambda^2)$. Moreover $\widehat{\eta}_P(t,\lambda)$ can be written in terms of L as follows:

$$\widehat{\eta}_P(t,\epsilon) = \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left((L\mathbb{D})e^{-t\lambda^2-\mathbb{D}^2}\right)d\lambda.$$

The variation in ϵ therefore becomes

$$\begin{aligned}
&\frac{d}{d\epsilon}\widehat{\eta}_P(t,\epsilon)\\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left((L\frac{d\mathbb{D}}{d\epsilon})e^{-t\lambda^2-\mathbb{D}^2}\right)d\lambda\\
&\;- \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left((L\mathbb{D})e^{-t\lambda^2}\int_0^1 e^{-u\mathbb{D}^2}(\frac{d\mathbb{D}}{d\epsilon}\mathbb{D}+\mathbb{D}\frac{d\mathbb{D}}{d\epsilon})e^{-(1-u)\mathbb{D}^2}\,du\right)d\lambda.
\end{aligned}$$

Integrating by parts and using the properties of L the first term on the right can be written as

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}(t^{\frac{1}{2}}\frac{d\mathbb{D}}{d\epsilon}e^{-t\lambda^2-\mathbb{D}^2})d\lambda\right)\\
&+\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{D}}{d\epsilon}e^{-t\lambda^2}\int_0^1 e^{-(1-u)\mathbb{D}^2}(\mathbb{D}\cdot L\mathbb{D}+L\mathbb{D}\cdot\mathbb{D})e^{-u\mathbb{D}^2}\,du\right)d\lambda.
\end{aligned}$$

Commuting $d\mathbb{D}/d\epsilon$ and $\exp(-(1-u)\mathbb{D}^2)$ to the right, and making use of the supercommutator property, give

$$\frac{d}{d\epsilon}\widehat{\eta}_P(t,\epsilon) = \frac{d}{dt}\left(\frac{1}{\pi}\int\limits_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}(t^{\frac12}\frac{d\mathbb{D}}{d\epsilon}e^{-t\lambda^2-\mathbb{D}^2})d\lambda\right)$$

$$+\frac{1}{\pi}\int\limits_{\mathbb{R}}\int\limits_0^1 \mathrm{STr}_{\mathrm{Cl}(1)}\left(e^{-t\lambda^2}(\mathbb{D}\cdot L\mathbb{D}+L\mathbb{D}\cdot\mathbb{D})e^{-u\mathbb{D}^2}\frac{d\mathbb{D}}{d\epsilon}e^{-(1-u)\mathbb{D}^2}\right)dud\lambda$$

$$-\frac{1}{\pi}\int\limits_{\mathbb{R}}\int\limits_0^1 \mathrm{STr}_{\mathrm{Cl}(1)}\left(e^{-t\lambda^2}L\mathbb{D}e^{-u\mathbb{D}^2}(\frac{d\mathbb{D}}{d\epsilon}\mathbb{D}+\mathbb{D}\frac{d\mathbb{D}}{d\epsilon})e^{-(1-u)\mathbb{D}^2}\right)dud\lambda.$$

Since $\mathbb{D}$ and $\exp(-u\mathbb{D}^2)$ commute, we can rewrite the sum of the second and third terms on the right as

$$\frac{1}{\pi}\int\limits_{\mathbb{R}}\int\limits_0^1 \mathrm{STr}\left[\mathbb{D}, \mathrm{e}^{-t\lambda^2}\mathrm{L}\mathbb{D}\mathrm{e}^{-\mathrm{u}\mathbb{D}^2}\frac{\mathrm{d}\mathbb{D}}{\mathrm{d}\epsilon}\mathrm{e}^{-(1-\mathrm{u})\mathbb{D}^2}\right]\mathrm{dud}\lambda.$$

Recalling that $\mathbb{D}=\mathbb{B}_t(\lambda,\epsilon)$ we finally obtain

$$\frac{d}{d\epsilon}\widehat{\eta}_P(t,\epsilon) = \frac{d}{dt}\left(\frac{1}{\pi}\int\limits_{\mathbb{R}} \mathrm{STr}_{\mathrm{Cl}(1)}\left(t^{\frac12}\frac{d\mathbb{B}_t(\lambda,\epsilon)}{d\epsilon}e^{-t\lambda^2-\mathbb{B}_t(\lambda,\epsilon)^2}\right)d\lambda\right)$$

(14.7)

$$+ d_B\alpha(t,\epsilon),$$

where

$$\alpha(t,\epsilon) = \frac{1}{\pi}\int\limits_{\mathbb{R}}\int\limits_0^1 \mathrm{STr}_{\mathrm{Cl}(1)}\left(e^{-t\lambda^2}L\mathbb{B}_t(\lambda,\epsilon)e^{-u\mathbb{B}_t(\lambda,\epsilon)^2}\right.$$

(14.8)

$$\left.\times\frac{d\mathbb{B}_t(\lambda,\epsilon)}{d\epsilon}e^{-(1-u)\mathbb{B}_t(\lambda,\epsilon)^2}\right)dud\lambda.$$

Hence (14.7) is precisely (14.6). q.e.d.

Corollary 3. *The difference* $\widehat{\eta}_P(\epsilon)-\widehat{\eta}_P$ *is exact.*

Proof. First observe that

$$\frac{d\mathbb{B}_t(\lambda,\epsilon)}{d\epsilon} = t^{\frac{1}{2}}\sigma\chi(t)\frac{d\widehat{\rho}_\epsilon}{d\epsilon}A_P^0,$$

which is a smoothing operator. This implies that

$$\frac{d\mathbb{B}_t(\lambda,\epsilon)}{d\epsilon}e^{-\mathbb{B}_t(\lambda,\epsilon)^2} \in \mathcal{C}^\infty([0,\infty),\Psi_{\partial\phi}^{-\infty}(\partial M,\mathbb{E}^0\otimes\mathbb{C}^2)),$$

and vanishes in $t<1$.

Since $\operatorname{null}(\eth_z^0+\widehat{\rho}_\epsilon(A_P^0)_z)=0$ for each $z\in B$, it follows that the heat kernel $\exp(-\mathbb{B}_t(\lambda,\epsilon)^2)$ is exponentially converging to zero as t tends to infinity. Thus integration in t of the first term on the right-hand side of (14.6) gives zero. On the other hand by dominated convergence and the definition of ρ_ϵ we have $\lim_{\epsilon\to 0}\widehat{\eta}_P(\epsilon)=\widehat{\eta}_P$. The corollary then follows by integration of formula (14.6) first in t then in ϵ. The integral in t is absolutely convergent, since $\alpha(t,\epsilon)$ is smooth, vanishes near $t=0$ and is exponentially decreasing as $t\to\infty$. q.e.d.

The eta form $\widehat{\eta}_P$ depends on the spectral section P through the regularizing finite rank operator A_P^0 of Lemma 8. We will show below that, up to an exact form, the definition of $\widehat{\eta}_P$ depends on only the spectral section P and not the particular choice of A.

Let A' be a different regularization and let $A(r), r\in[0,1]$, be a smooth homotopy through operators as in Lemma 8. Consider the family of Cl(1)-superconnections

$$\widetilde{\mathbb{B}}_t(r) = t^{\frac{1}{2}}(\sigma\eth^0+\sigma\chi(t)A(r)) + \mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]}$$

and the differential forms

$$\widehat{\eta}_P(t,r) = \frac{1}{\sqrt{\pi}}\operatorname{STr}_{\mathrm{Cl}(1)}\left(\frac{d\widetilde{\mathbb{B}}_t(r)}{dt}e^{-\widetilde{\mathbb{B}}_t(r)^2}\right). \tag{14.9}$$

First we have the analogue of Proposition 13:

Proposition 14. *The differential forms $\widehat{\eta}_P(t,r)$ depend smoothly on r and*

$$\begin{aligned}\frac{d}{dr}\widehat{\eta}_P(t,r) = &\frac{d}{dt}\left(\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}\operatorname{STr}_{\mathrm{Cl}(1)}(\sigma\chi(t)\frac{dA(r)}{dr}e^{-\widetilde{\mathbb{B}}_t(r)^2})\right)\\ &+ d_B\beta(t,r).\end{aligned} \tag{14.10}$$

Proof. Taking the derivative of (14.9) with respect to r, using Duhamel's formula, the vanishing of the supertrace on supercommutators and the fact that $\left[\widetilde{\mathbb{B}}_t(r), e^{-u\widetilde{\mathbb{B}}_t(r)^2}\right] = 0$ we obtain

$$\begin{aligned}
&\frac{d}{dr}\widehat{\eta}_P(t,r) \\
&\quad = \frac{1}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\left(\frac{d}{dt}t^{\frac{1}{2}}\sigma\chi(t)\right)\frac{dA(r)}{dr}e^{-\widetilde{\mathbb{B}}_t(r)^2}\right) \\
&\qquad - \frac{1}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\int_0^1 t^{\frac{1}{2}}\sigma\chi(t)\frac{dA(r)}{dr}e^{-u\widetilde{\mathbb{B}}_t(r)^2}\frac{d\widetilde{\mathbb{B}}_t(r)^2}{dt}e^{-(1-u)\widetilde{\mathbb{B}}_t(r)^2}du\right) \\
&\qquad - \frac{1}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}\left(\int_0^1 t^{\frac{1}{2}}\left[\widetilde{\mathbb{B}}_t(r), e^{-u\widetilde{\mathbb{B}}_t(r)^2}\sigma\chi(t)\right.\right. \\
&\qquad\qquad \left.\left.\times\frac{dA(r)}{dr}e^{-(1-u)\widetilde{\mathbb{B}}_t(r)^2}\frac{d\widetilde{\mathbb{B}}_t(r)}{dt}\right]du\right).
\end{aligned}$$

The right-hand side can be rewritten as

$$\frac{d}{dt}\left(\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}\mathrm{STr}_{\mathrm{Cl}(1)}(\sigma\chi(t)\frac{dA(r)}{dr}e^{-\widetilde{\mathbb{B}}_t(r)^2})\right) + d_B\beta(t,r),$$

where

$$\begin{aligned}
\beta(t,r) = -\,\mathrm{STr}_{\mathrm{Cl}(1)}\left(\int_0^1 \frac{t^{\frac{1}{2}}}{\sqrt{\pi}}e^{-u\widetilde{\mathbb{B}}_t(r)^2}\sigma\chi(t)\frac{dA(r)}{dr}\right. \\
\left.\times e^{-(1-u)\widetilde{\mathbb{B}}_t(r)^2}\frac{d\widetilde{\mathbb{B}}_t(r)}{dt}du\right),
\end{aligned} \tag{14.11}$$

which gives (14.10). q.e.d.

Integration in t and in r of formula (14.10) yields

Corollary 4. *The eta form $\widehat{\eta}_P$ is well defined, modulo an exact form, by the choice of a spectral section P for $\eth^0$.*

15. Family index theorem

The main formula, (2), in the Introduction

$$\mathrm{Ch}(\mathrm{Ind}_P(\eth)) = \phi_*(\mathrm{AS}) - \frac{1}{2}\widehat{\eta}_P, \quad \text{in } H^*(B) \tag{15.1}$$

can now be proved.

By Proposition 6 we have

$$\mathrm{Ind}_P(\eth) = \mathrm{Ind}_0(\eth + A_P(\epsilon)), \quad \text{in } K^0(B) \tag{15.2}$$

with

$$\mathrm{Ind}_0(\eth + A_P(\epsilon)) = [\mathrm{null}_0(\eth^+ + A_P^+)] \ominus [\mathrm{null}_0(\eth^- + A_P^-)], \tag{15.3}$$

the subscript referring to the fact that these are the null spaces on the metric (or b-) L^2 spaces

$$\eth_z^\pm + (A_P^\pm)_z : H_b^1(M_z; E^\pm) \longrightarrow L_b^2(M_z; E^\mp). \tag{15.4}$$

In order to give a formula for the Chern character of $\mathrm{Ind}_0(\eth + A_P(\epsilon))$ we shall initially assume the dimensions of the null spaces of (15.4) to be constant. Under this additional hypothesis the index bundle (15.3) is a smooth $\mathbb{Z}_2$-graded vector bundle.

Let Π_0 be the family of orthogonal projections onto the null spaces of $\eth + A_P(\epsilon)$ acting as in (15.4). Since this family is self-adjoint, Π_0 is also the family of projections onto the null spaces of $(\eth + A_P(\epsilon))^2$. Let G be the family of generalized inverses of $(\eth + A_P(\epsilon))^2$. By construction,

$$0 \notin \mathrm{Im}\,\mathrm{spec}_b(\eth_z + A_P(\epsilon)_z)^2 \quad \forall\ z \in B,$$

so from Corollary 5 in the Appendix it follows that

$$\Pi_0 \in \Psi_\phi^{-\infty,\delta}(M;E) \text{ and } G \in \Psi_{b,\phi}^{-2,\delta}(M;E) \tag{15.5}$$

for some $\delta > 0$.

Now consider the rescaled Bismut superconnection

$$\mathbb{A}_t(\epsilon) = t^{\frac{1}{2}}(\eth + \chi(t)A_P(\epsilon)) + \mathbb{A}_{[1]} + t^{-\frac{1}{2}}\mathbb{A}_{[2]}$$

adapted to the family $(\eth + A_P(\epsilon))$. The operator

$$\nabla^{\mathrm{null}} = \Pi_0 \mathbb{A}_{[1]} \Pi_0 \tag{15.6}$$

is a connection on the smooth $\mathbb{Z}_2$-graded vector bundle

$$\mathrm{Ind}_0 \equiv \mathrm{Ind}_0(\eth + A_P(\epsilon)),$$

and we consider its Chern character

$$\mathrm{Ch}(\mathrm{Ind}_0, \nabla^{\mathrm{null}}) = \mathrm{STr}(\mathrm{e}^{-(\nabla^{\mathrm{null}})^2}) \in \mathcal{C}^\infty(\mathrm{B}, \Lambda^*\mathrm{B}). \tag{15.7}$$

Proposition 15. *If* $\mathrm{Ind}_0(\eth + A_P(\epsilon))$ *is a smooth vector bundle, the limit*

$$\lim_{t\to\infty} b\text{-}\mathrm{Ch}(\mathbb{A}_t(\epsilon)) = \mathrm{Ch}(\mathrm{Ind}_0, \nabla^{\mathrm{null}})$$

holds with respect to each C^l*-norm on* B.

Proof. For each $t > 0$ the heat kernel $\exp(-\mathbb{A}_t(\epsilon)^2)$ is an element in $\Psi^{-\infty}_{b,\phi}(M;\mathbb{E})$. Following the proof given by Berline and Vergne in the boundaryless case ([12], [13]) consider for each $\delta > 0$ the spaces of operators

$$\mathcal{M}^\delta(B) = \Psi^{*,\delta}_{b,\phi}(M;\mathbb{E}) \text{ and } \mathcal{N}^\delta(B) = \Psi^{-\infty,\delta}_{\phi}(M;\mathbb{E}). \tag{15.8}$$

We shall take δ as in (15.5) and then by Theorem 4 the family G of generalized inverses of $(\eth + A_P(\epsilon))^2$ acts on the left and on the right on $\mathcal{N}^\delta(B)$. Writing $\mathbb{E}^{(i)} = \phi^*\Lambda^i B \otimes E$ consider the filtration

$$\mathcal{M}^\delta_i(B) = \sum_{j\geq i} \Psi^{*,\delta}_{b,\phi}(M;\mathbb{E}^{(j)})$$

and the analogous one for $\mathcal{N}^\delta(B)$.

The discussion in [20, Chapter 7] on the large time behaviour of the heat kernel of the square of a Dirac operator extends immediately to the family $(\eth + A_P(\epsilon))^2$, under the assumption that the small eigenvalues are of constant rank, and hence smooth with smooth eigenspaces. This gives, for each $z \in B$, a decomposition

$$e^{-t(\eth_z + A_P(\epsilon)_z)^2} = \sum_{\lambda_i(z)<\delta} e^{-t\lambda_i(z)}P_i(z) + R_\infty(t,z) \tag{15.9}$$

with the $\lambda_i(z)$ the small eigenvalues of $(\eth_z + A_P(\epsilon)_z)^2$, and $P_i(z)$ the associated eigenprojections. Moreover the remainder term is such that for some $\delta' > 0$ $\exp(t\delta')R_\infty(t)$ is uniformly bounded, with all its t-derivatives, with values in $\Psi^{-\infty,\delta}_{\phi}(M,E)$. Thus if $Q = \mathrm{Id} - P_0$ it follows that for t large the kernel of $Q\exp(-t(\eth + A_P(\epsilon))^2)Q$ is exponentially decreasing as a function of t with values in $\Psi^{-\infty,\delta}_{b,\phi}(M,E)$.

Let $\mathcal{F} = \mathbb{A}(\epsilon)^2$ be the curvature of the superconnection and consider the decomposition

$$\mathcal{F} = \begin{pmatrix} P_0\mathcal{F}P_0 & P_0\mathcal{F}Q \\ Q\mathcal{F}P_0 & Q\mathcal{F}Q \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}, \tag{15.10}$$

where, again by (15.5) and Theorem 4, $X, Y, Z \in \mathcal{N}^\delta(B)$. By applying the diagonalization lemma of Berline and Vergne we can find an element $g \in \mathcal{M}^\delta(B)$, of the form $g = 1 + K$ with $K \in \mathcal{N}^\delta_1(B)$ such that

$$\mathcal{F} = g^{-1}\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} g \text{ with}$$

$$U = (\nabla^{\text{null}})^2 + U^+,\ U^+ \in \mathcal{N}^\delta_3(B) \text{ and} \tag{15.11}$$

$$V = Q((\eth + A_P(\epsilon))^2)Q + V^+,\ V^+ \in \mathcal{N}^\delta_1(B).$$

Here the fact that G acts on the right and on the left of $\mathcal{N}^\delta(B)$ has been used. This decomposition, the large time behaviour of the heat kernel of $\exp(-tQ((\eth + A_P(\epsilon))^2)Q)$ discussed above and the analogue of the Volterra series (11.2) for the heat kernels of U and V together show, as in [13], that

$$b\text{-STr}(e^{-\mathbb{A}_t^2(\epsilon)}) = b\text{-STr}(e^{-(\nabla^{\text{null}})^2}) + O(t^{-\frac{1}{2}}).$$

Since Π_0 is a family of finite rank, and thus trace class, and operators on L^2_b, it follows that

$$b\text{-STr}e^{-(\nabla^{\text{null}})^2} = \text{STr}e^{-(\nabla^{\text{null}})^2}.$$

By taking the limit as $t \to \infty$ the proposition follows for the supremum norm on the base. Using Duhamel's formula the same type of argument gives the convergence in each C^l. q.e.d.

Next we analyze the behaviour of $b\text{-Ch}(\mathbb{A}_t(\epsilon))$ as $t \downarrow 0$. First consider the rescaled, unperturbed Bismut superconnection $\mathbb{A}_t$. By definition

$$b\text{-Ch}(\mathbb{A}_t)(z) = {}^{\nu}\!\!\int_{M_z} \text{str}_E(e^{-\mathbb{A}_t^2})(x,x) \tag{15.12}$$

with $\phi(x) = z$. With δ_t the automorphism of $\mathcal{C}^\infty(B, \Lambda^* B)$ which acts on $\mathcal{C}^\infty(B, \Lambda^i B)$ by multiplication by $t^{-\frac{i}{2}}$ we can rewrite (15.12) as

$$b\text{-Ch}(\mathbb{A}_t)(z) = {}^{\nu}\!\!\int_{M_z} \delta_t(\operatorname{str}_E(e^{-t\mathbb{A}^2})(x,x)). \tag{15.13}$$

Let $\widetilde{\Delta}_{h,\phi}$ be the lift of the submanifold $[0,\infty) \times \Delta_{b,\phi} \subset [0,\infty) \times M^2_{b,\phi}$ to the heat space $M^2_{\eta,\phi}$. Directly from the definition of the heat calculus the kernel of $\exp(-t\mathbb{A}^2)$ restricted to $\widetilde{\Delta}_{h,\phi}$ must have an asymptotic expansion

$$e^{-t\mathbb{A}^2} \restriction \widetilde{\Delta}_{h,\phi} \sim \sum_{j=0}^{\infty} A_j t^{-\frac{n}{2}+j} \quad \text{as } t \downarrow 0,$$

where n is the dimension of the fibres of $\phi : M \longrightarrow B$, and the coefficients A_j are elements of $\mathcal{C}^\infty(M; \phi^*\Lambda^* B \otimes \hom(E) \otimes {}^b\Omega_{\text{fib}})$ under the identification $\Delta_{b,\phi} \leftrightarrow M$.

To investigate the asymptotic expansion of

$$\delta_t(\operatorname{str}_E(e^{-t\mathbb{A}^2}) \restriction \widetilde{\Delta}_{h,\phi}) \in t^{-\frac{n}{2}}\mathcal{C}^\infty([0,\infty) \times M; \phi^*\Lambda^* B \otimes {}^b\Omega_{\text{fib}})$$

recall formula (9.25) for the curvature $\mathbb{A}^2$ of the Bismut superconnection:

$$\mathbb{A}^2 = \Delta^{M/B} + \frac{1}{4}S_{M/B} - \frac{1}{2}\sum_{a,b} m_0(e^a)m_0(e^b)K'_E(e^a, e^b).$$

In the boundaryless case this generalized Lichnerowicz formula allows the rescaling argument of Getzler ([15]) to be applied fiberwise to the family of heat kernels $\exp(-t\mathbb{A}^2)$. In [13] this method is used to prove Bismut's theorem, i.e., to show that

$$\delta_t(\operatorname{str}_E(e^{-t\mathbb{A}^2}) \restriction \widetilde{\Delta}_{h,\phi}) \in \mathcal{C}^\infty([0,\infty) \times M; \phi^*\Lambda^* B \otimes {}^b\Omega_{\text{fib}})$$

and to compute its value at $t = 0$ in terms of the $\widehat{A}$-genus of the vertical tangent bundle and the twisted Chern character of the bundle E. As explained in [20] Getzler's rescaling carries over to the b-setting unchanged.

To state the analogue of Bismut's theorem recall from Section 9 the connection $\nabla^{M/B}$ on the vertical b-tangent bundle ${}^bT(M/B)$. Formula (9.10) implies that the associated curvature tensor is an ordinary smooth form, $R^{M/B} \in \mathcal{C}^\infty(M; \Lambda^2 M \otimes \hom({}^bT(M/B)))$. Let $\widehat{A}(M/B)$ be the associated $\widehat{A}$-genus

$$\widehat{A}(M/B) = \det^{\frac{1}{2}}\left(\frac{R^{M/B}/2}{\sinh(R^{M/B}/2)}\right) \in \mathcal{C}^\infty(M; \Lambda^* M),$$

where we follow the convention of [13]. Let K_E' be the twisting curvature of the bundle E introduced in Section 9. As this curvature is computed in terms of the given fibre connection on E, assumption (1.9) implies that $K_E' \in \mathcal{C}^\infty(M; \Lambda^2 M \otimes \hom_{\mathrm{Cl}_\phi}(E))$. Denote by str' the supertrace induced on $\hom_{\mathrm{Cl}_\phi}(E)$ and let

$$\mathrm{Ch}'(E) = \mathrm{str}'(\exp(-K_E')) \in \mathcal{C}^\infty(M; \Lambda^* M)$$

be the twisted Chern character of the bundle E. Finally let $\mathrm{Ev}(M/B)_n$ be the evaluation of a differential form on M to its component of maximal vertical degree according to the decomposition $\Lambda^* M = \phi^* \Lambda^* B \otimes \Lambda^*(M/B)$, so

$$\mathrm{Ev}(M/B)_n : \mathcal{C}^\infty(M; \Lambda^* M) \longrightarrow \mathcal{C}^\infty(M; \phi^* \Lambda^* B \otimes \Lambda^n(M/B).$$

We can now state the extension of Bismut's theorem to the b-setting. Let $\mathbb{A}$ be the Bismut superconnection introduced in Section 9.

Proposition 16. *The section $\delta_t(\mathrm{str}_E(e^{-t\mathbb{A}^2})$ restricts to $\widetilde{\Delta}_{h,\phi}$ to an element of the space $\mathcal{C}^\infty([0,\infty) \times M; \phi^* \Lambda^* B \otimes {}^b\Omega_{\mathrm{fib}})$ and has a limit as $t \downarrow 0$ equal to*

$$\mathrm{Ev}(M/B)_n \left(\frac{1}{(2\pi i)^{\frac{n}{2}}} \widehat{A}(M/B)\,\mathrm{Ch}'(E)\right), \tag{15.14}$$

hence the differential form b-$\mathrm{Ch}(\mathbb{A}_t)$ has the limit as $t \downarrow 0$

$$\lim_{t\downarrow 0} b\text{-}\mathrm{Ch}(\mathbb{A}_t) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E). \tag{15.15}$$

Since the perturbation, $\mathbb{A}_t(\epsilon) - \mathbb{A}_t$, vanishes near $t = 0$ it also follows that

$$\lim_{t\downarrow 0} b\text{-}\mathrm{Ch}(\mathbb{A}_t(\epsilon)) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E). \tag{15.16}$$

If we drop assumption (1.9) and consider an arbitrary Clifford Hermitian fibre b-connection on E, then $\mathrm{Ch}'(E) \in \mathcal{C}^\infty(M; {}^b\Lambda^* M)$ and the integral on the right-hand side of (15.15) must be regularized as in (12.1).

This result can be used as in [13] to show that

$$b\text{-STr}\left(\frac{d\mathbb{A}_t(\epsilon)}{dt}e^{-\mathbb{A}_t(\epsilon)^2}\right) = O(t^{-\frac{1}{2}}) \quad \text{as } t \downarrow 0. \tag{15.17}$$

Similarly, if the dimensions of the null spaces in (15.4) do not vary, Proposition 15 and its proof imply that

$$b\text{-STr}\left(\frac{d\mathbb{A}_t(\epsilon)}{dt}e^{-\mathbb{A}_t(\epsilon)^2}\right) = O(t^{-\frac{3}{2}}) \quad \text{as } t \uparrow \infty. \tag{15.18}$$

Thus, under the assumption that $\dim(\text{null}(\eth_z + A_P(\epsilon)_z))$ is constant in $z \in B$, integration of formula (13.21) from 0 to ∞ and the application of Corollary 3, (15.16) and (15.5), give the fundamental formula

$$\begin{aligned} \text{Ch}(\text{Ind}_0(\eth + A_P(\epsilon))) = & \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M/B} \widehat{A}(M/B)\,\text{Ch}'(E) \\ & - \frac{1}{2}\widehat{\eta}_P - d_B\alpha(\epsilon) - d_B\gamma(\epsilon) \end{aligned} \tag{15.19}$$

with $\alpha(\epsilon)$ obtained by integration in ϵ and in $t \in [0, \infty)$ of (14.18) and

$$\gamma(\epsilon) = \int_0^\infty b\text{-STr}\left(\frac{d\mathbb{A}_t(\epsilon)}{dt}e^{-\mathbb{A}_t(\epsilon)^2}\right)dt.$$

In the general case where $\dim(\text{null}(\eth_z^+ + A_P^+(\epsilon)_z))$ varies, we use the regularization of the index bundle in Section 6. Thus there exist $N \in \mathbb{N}$ and a smooth family of operators, $G_z : \mathbb{C}^N \longrightarrow \mathcal{C}_c^\infty(\mathring{M}_z; E^-)$, with the property that

$$(\eth_{P,z}^+ + A_P^+(\epsilon)_z) \oplus G_z : H_b^1(M_z; E^+) \oplus \mathbb{C}^N \longrightarrow L_b^2(M_z; E^-)$$

is surjective for each $z \in B$. We can apply (13.21) to the rescaled, perturbed Bismut superconnection

$$\begin{aligned} \widetilde{\mathbb{A}}(\epsilon) = & t^{\frac{1}{2}} \begin{pmatrix} 0 & ((\eth^+ + \chi(t)A_P^+(\epsilon)) \oplus \chi(t)G)^* \\ (\eth^+ + \chi(t)A_P^+(\epsilon)) \oplus \chi(t)G & 0 \end{pmatrix} \\ & + \mathbb{A}_{[1]} + t^{-\frac{1}{2}}\mathbb{A}_{[2]}. \end{aligned}$$

Since $\mathrm{null}_0((\eth^+ + A_P^+(\epsilon)) \oplus G)$ is a vector bundle, Proposition 15 shows that %beginequation

$$\lim_{t\to\infty} b\text{-}\mathrm{Ch}(\widetilde{\mathbb{A}_t}(\epsilon)) = \mathrm{Ch}(\mathrm{null}_0((\eth^+ + A_P^+(\epsilon)) \oplus G), \nabla^{\mathrm{null}}), \tag{15.20}$$

whereas the analogue of (15.16) gives

$$\lim_{t\downarrow 0} b\text{-}\mathrm{Ch}(\widetilde{\mathbb{A}_t}(\epsilon)) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E) + N. \tag{15.21}$$

On the other hand since G has values in sections vanishing to infinite order at the boundary, there is no change in the boundary contribution in formula (13.21). Recall that, by definition,

$$\begin{aligned} &\mathrm{Ind}_0(\eth + A_P(\epsilon)) \\ &\quad = [\mathrm{null}_0(\eth^+ + A_P^+(\epsilon)) \oplus G] \ominus [B \times \mathbb{C}^N] \\ &\qquad \text{in } K^0(B). \end{aligned} \tag{15.22}$$

Thus by Corollary 3, (15.20), (15.21) and the above remarks we obtain the main result of this paper

Theorem 1. *Let $\eth \in \mathrm{Diff}^1_{b,\phi}(M;E)$ be a family of generalized Dirac operators on manifolds with boundary as in Section 1 and let P be a spectral section for the boundary family $\eth^0$. If $\mathrm{Ind}_P(\eth) \in K^0(B)$ is the index bundle associated to the family of generalized Atiyah-Patodi-Singer boundary problems defined by P as in Section 6, then the following formula holds*

$$\begin{aligned} \mathrm{Ch}(\mathrm{Ind}_P(\eth)) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{M/B} \widehat{A}(M/B)\,\mathrm{Ch}'(E) - \frac{1}{2}\widehat{\eta}_P \\ \text{in } H^*(B) \end{aligned} \tag{15.23}$$

where $\widehat{\eta}_P$ is the eta form defined in Section 14.

Returning to the incomplete context of Section 3 we deduce

Theorem 2. *Let $\widehat{\eth} \in \mathrm{Diff}^1_{b,\phi}(\widehat{M};E)$ be a family of generalized Dirac operators on manifolds with boundary as in Section 3 and let P be a spectral section for the boundary family $\eth^0$. If $\mathrm{Ind}_P(\eth) \in K^0(B)$ is the index bundle associated to the family of generalized Atiyah-Patodi-Singer boundary problems defined by P as in Section 3, then the following formula holds:*

$$\begin{aligned} \mathrm{Ch}(\mathrm{Ind}_P(\widehat{\eth})) = \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{\widehat{M}/B} \widehat{A}(\widehat{M}/B)\,\mathrm{Ch}'(E) - \frac{1}{2}\widehat{\eta}_P \\ \text{in } H^*(B), \end{aligned} \tag{15.24}$$

where $\widehat{\eta}_P$ is the eta form defined in Section 14.

Proof. As already noted the index bundle over B can be realized as the index bundle for the corresponding complete problem, with cylindrical ends added, to which Theorem 1 applies. To deduce (15.24) from (15.23) it is only necessary to see that the part of the integral in (15.23) over the cylindrical end vanishes identically. This is a consequence of the local $\mathbb{R}^+$-invariance. q.e.d.

16. Jumps of $\widehat{\eta}_P$

The form $\widehat{\eta}_P$, defined by a family of Dirac operators on the odd-dimensional fibres of a fibration $\psi : M' \longrightarrow B$ and a regularization determined by a choice of spectral section, is well-defined up to an exact form as follows from Proposition 13. It is a consequence of the relative family index theorem that, assuming the family of Dirac operators arises as the boundary family from an even-dimensional fibration of manifolds with boundary, the difference $\widehat{\eta}_{P_1} - \widehat{\eta}_{P_2}$ for two spectral sections is a closed form representing in cohomolgy the Chern character of the difference bundle $2[P_1 - P_2]$. We shall show how this result can be deduced in general, without any cobordism assumption.

Proposition 17. *If $\eth^0$ is a family of Dirac operators arising from a bundle of Clifford modules with Hermitian Clifford connection on the fibres of a fibration, $\psi : M' \longrightarrow B$, with odd-dimensional, compact boundaryless fibres, and the index family of $\eth^0$ vanishes in $K^1(B)$, then for any two spectral sections P_1 and P_2 of $\eth^0$*

$$\widehat{\eta}_{P_1} - \widehat{\eta}_{P_2} = 2\,\mathrm{Ch}([P_1 - P_2]) \text{ in } H^*(B). \tag{16.1}$$

Proof. By Proposition 13 it suffices to consider any regularizations of $\eth^0$ to an invertible family defined by the spectral sections P_1 and P_2. In fact the linearity of (16.1) and Proposition 2 mean that it is enough to suppose that P_1 is the identity on the range of P_2, so in particular P_1 and P_2 commute. Consider a regularization of $\eth^0$, relative to P_1, as constructed in Lemma 8. In that construction it can be assumed that s is such that P_2, and hence P_1, acts as the identity on all eigenspaces of $\eth^0$ with eigenvalues greater than, or equal to, s. Then (8.2) holds with

$P = P_1$ and $P = P_2$ for the same Q and R. Let

$$\begin{aligned}\widetilde{D}_i =& Q \circ \eth^0 \circ Q + sP_i R(\operatorname{Id} - Q) \\ & + (\operatorname{Id} - R) \circ \eth^0 \circ (\operatorname{Id} - R) \\ & - s(\operatorname{Id} - P_i) R(\operatorname{Id} - Q),\ \ i = 1, 2,\end{aligned} \tag{16.2}$$

be the corresponding regularizations given by (8.3). Then

$$\widetilde{D}_1 = \widetilde{D}_2 + 2s(P_1 - P_2)R(\operatorname{Id} - Q). \tag{16.3}$$

Here $(P_1 - P_2)R(\operatorname{Id} - Q) = P_1 - P_2$ is a self-adjoint projection of finite rank commuting with both $\widetilde{D}_1$ and $\widetilde{D}_2$, which are both invertible families. Thus the homotopy

$$\widetilde{D}_r = \frac{1}{2}(1 + r)\widetilde{D}_1 + \frac{1}{2}(1 - r)\widetilde{D}_2,\ \ r \in [-1, 1] \tag{16.4}$$

consists of invertible operators for $r \neq 0$ with fixed eigenvalues except for a fixed, finite dimensional, eigenspace with variable eigenvalue sr. Let $\Pi_0 = P_1 - P_2$ be the finite rank smoothing operator giving the orthogonal projection onto this eigenspace, so if $\Pi_1 = \operatorname{Id} - \Pi_0$ then

$$\Pi_1 \widetilde{D}_r \Pi_1 \text{ is independent of } r. \tag{16.5}$$

Thus it suffices to consider the variation of the form in (14.5) as r changes from -1 to 1. From Proposition 14 it varies smoothly, and exactly, except possibly at $r = 0$. We shall show that the jump across $r = 0$ gives twice the Chern character for the bundle which is the range of the projection $P_1(\operatorname{Id} - P_2)$. To do so we follow (with the simplification that there is no boundary) and slightly modify, the discussion in Section 15 in turn following [12] (see also [13, Chapter 9]), of the long-time behaviour of the Chern character of a rescaled superconnection adapted to a family of operators with null space of constant rank.

Let $\psi' : M \times I \longrightarrow B \times I$ be the product fibration with additional parameter $r \in I \subset [-1, 1]$. We will take I to be an interval containing 0. With $B' = B \times [-1, 1]$, let $\mathbb{E}^{(j)}$ denote now the lift to $M \times I$ of the bundle from M (so only with coefficient forms from B) and consider the filtration

$$\mathcal{N}_i(B') = \sum_{j \geq i} \Psi^{-\infty}_{\psi'}(M'; \mathbb{E}^{(j)})$$

of the space of smooth families of smoothing operators on the fibres which raise the form degree by at least i. Then if $\mathcal{M}(B')$ is the space of all smooth families of pseudodifferential operators on the fibres with similar filtration $\sum_{j\geq i}\Psi^*_{\psi'}(B';\mathbb{E}^{(j)})$, the $\mathcal{N}_i(B')\subset\mathcal{M}_i(B')$ are two-sided ideals over $\mathcal{M}(B')$ which satisfy

$$\begin{aligned}\mathcal{M}_i(B')\cap\mathcal{N}_0(B') = &\ \mathcal{N}_i(B'),\\ \mathcal{M}_i(B')\circ\mathcal{M}_j(B')\subset &\ \mathcal{M}_{i+j}(B')\text{ and}\\ \mathcal{M}_i(B') = &\ \{0\},\ i>\dim B.\end{aligned}\tag{16.6}$$

If $\mathbb{B}$ is the Cl(1)-superconnection adapted to $\eth^0$, the decomposition of the curvature, $\mathcal{F}(r)=\mathbb{B}(r)^2$, of the Cl(1)-superconnection $\mathbb{B}(r)=\mathbb{B}+\sigma(\frac12(1+r)\widetilde{D}_1+\frac12(1-r)\widetilde{D}_2-\eth^0)$ as in (15.10) is

$$\mathcal{F}(r)=\begin{pmatrix}X & Y\\ Z & T\end{pmatrix},\tag{16.7}$$

where now

$$\begin{aligned}&X-r^2s^2\,\mathrm{Id}\in\mathcal{N}_2(B'),\ Y,Z\in\mathcal{N}_1(B')\\ &\text{and } T_{[0]}\in\mathcal{M}_0(B')\text{ is invertible.}\end{aligned}\tag{16.8}$$

We shall consider successive conjugating matrices

$$g_{(i)}(r)=g_ig_{i-1}\cdots g_1,\ g_i=\mathrm{Id}+\begin{pmatrix}0 & A_i(r)\\ B_i(r) & 0\end{pmatrix},\quad i=1,\dots,\dim B\tag{16.9}$$

with entries in $\mathcal{N}_i(B')$. We wish to arrange that

$$g_{(i)}(r)\mathcal{F}(r)g_{(i)}^{-1}(r)=\begin{pmatrix}X_{(i+1)} & Y_{(i+1)}\\ Z_{(i+1)} & T_{(i+1)}\end{pmatrix}\tag{16.10}$$

with $Y_{(i+1)}$, $Z_{(i+1)}\in\mathcal{M}_{i+1}(B')$. Assuming this has been done at level $i-1$, $g_i(r)$ is to be chosen so that

$$g_i(r)\begin{pmatrix}X_{(i)} & Y_{(i)}\\ Z_{(i)} & T_{(i)}\end{pmatrix}g_i^{-1}(r)=\begin{pmatrix}X_{(i+1)} & Y_{(i+1)}\\ Z_{(i+1)} & T_{(i+1)}\end{pmatrix}\tag{16.11}$$

$$\text{with } X-X_{(i)},\ T-T_{(i)}\in\mathcal{M}_1(B').\tag{16.12}$$

Now (16.9) shows that $g_i^{-1}(r)=\mathrm{Id}-\begin{pmatrix}0 & A_i(r)\\ B_i(r) & 0\end{pmatrix}$ up to terms in $\mathcal{N}_{2i}(B')$, which can be absorbed on the right in (16.11). Moreover

(16.12) holds, and ignoring similar higher order terms $A_i(r)$ and $B_i(r)$ need only be chosen to satisfy

$$\begin{aligned} T_{[0]}B_i - r^2s^2B_i &= Z_{(i)}, \\ A_iT_{[0]} - r^2s^2A_i &= -Y_{(i)}, \end{aligned} \tag{16.13}$$

the zero-form components of $X_{(i)}$ and $T_{(i)}$ being independent of i. As $T_{[0]}$ is invertible, (16.13) can certainly be solved for small r. We also see inductively that (16.12) can be extended to conclude that

$$\begin{aligned} X_{(i)} - X + Y_i \cdot G \cdot Z_i \in r^3\mathcal{M}_2(B') + \mathcal{M}_3(B'), \\ G = T_{[0]}^{-1}. \end{aligned} \tag{16.14}$$

Thus the diagonalization procedure of Berline and Vergne reduces the curvature to a form similar to (15.11):

$$\mathcal{F} = g^{-1}\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} g \tag{16.15}$$

with

$$\begin{aligned} U &= r^2s^2 + (\nabla^{\mathrm{null}})^2 + U^+, \\ U^+ &\in \mathcal{N}_3(B') + r^3\mathcal{N}_2(B'); \\ V &= \Pi_1((\widetilde{D}_r)^2)\Pi_1 + V^+, \\ V^+ &\in \mathcal{N}_1(B'), \end{aligned} \tag{16.16}$$

where (16.5) should also be recalled.

The formula, (14.10), for the variation of the eta form gives

$$\begin{aligned} \widehat{\eta}_{P_1} - \widehat{\eta}_{P_2} &= \int_0^\infty \int_{-1}^1 \frac{d\widehat{\eta}(t,r)}{dr} dr dt \\ &= \lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \int_{-1}^1 t^{\frac{1}{2}} \operatorname{STr}_{\mathrm{Cl}(1)} \left(\sigma \frac{dA(r)}{dr} e^{-\mathbb{B}_t^2(r)} \right) dr \\ &\quad + d_B \int_0^\infty \int_{-1}^1 \beta(t,r) dr dt, \end{aligned} \tag{16.17}$$

where we have used the fact that $\mathrm{STr}_{\mathrm{Cl}(1)}\, d\widehat{\eta}(t,r)/dr$ vanishes in $t<1$, $\beta(t,r)$ being given by (11.9). From (16.3) and (16.4) the limit in (16.17) can be written as

$$\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}2s\int_{-1}^{1} t^{\frac12}\,\mathrm{Tr}\left(\Pi_0 \mathrm{e}^{-\mathbb{B}_t^2(\mathrm{r})}\right)\mathrm{dr}.$$

Let δ_t be the automorphism introduced in the previous section and set $g_t=\delta_t g$. By (16.15) and (16.16) the heat kernel of the rescaled superconnection is such that

$$g_t e^{-\mathbb{B}_t^2(r)} g_t^{-1}$$

is diagonal with the Π_1 term uniformly exponentially decreasing as $t\to\infty$. Thus the limit can be replaced by

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}2s\int_{-1}^{1} t^{\frac12}\,\mathrm{tr}\Big(&\Pi_0 g_t^{-1}\Pi_0\exp(-tr^2s^2-(\nabla^{\mathrm{null}})^2\\ &+r^3H_1+t^{-\frac12}H_2)\Pi_0 g_t\Pi_0\Big)\,dr,\end{aligned} \tag{16.18}$$

where the trace is now finite dimensional, which is the image of Π_0 on the bundle. Using the analogue of the Volterra series (11.2), setting $R=t^{\frac12}sr$ and noting that $\Pi_0 g_t\Pi_0=\mathrm{Id}+O(t^{-\frac12})$ and similarly for its inverse, the limit evaluates to

$$\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}2\int_{-st^{\frac12}}^{st^{\frac12}}\mathrm{tr}\exp\left(-R^2-(\nabla^{\mathrm{null}})^2\right)dR.$$

This is just twice the Chern character of the bundle, for the projected connection ∇^{null}. A similar analysis confirms that the exact term in (16.17) converges in $\mathcal{C}^\infty(B;\Lambda^*)$ so (16.1) holds. q.e.d.

One application of such a 'jump formula' is to the case of a family of Dirac operators as in Section 3 where the boundary family has null space of constant rank. Then the spectral section can be chosen as $P=P_+$, the projection onto the eigenspaces of the boundary family with non-negative eigenvalues. In this case the unregularized eta form

$$\widehat{\eta}=\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{STr}_{\mathrm{Cl}(1)}\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right)dt$$

converges, as is easily seen using the rescaling described above. Following the proof of Proposition 17 we find that

$$\widehat{\eta}_{P_+} - \widehat{\eta} = \mathrm{Ch}(\mathrm{null}(\eth^0)). \tag{16.19}$$

Inserting this in (15.24) gives the formula of Bismut and Cheeger from [11]:

Theorem 3. *Let* $\widehat{\eth} \in \mathrm{Diff}^1_{b,\phi}(\widehat{M}; E)$ *be a family of generalized Dirac operators on manifolds with boundary as in Section 3, and suppose that the family of self-adjoint Dirac operators induced on the boundary,* $\eth^0 \in \mathrm{Diff}^1_{\partial\phi}(\partial\widehat{M}, E_{\partial\widehat{M}})$, *has null space of constant rank. If* $\mathrm{Ind}(\eth) \in K^0(B)$ *is the index bundle associated to the family of boundary problems defined by* P_+, *then we have the Atiyah-Patodi-Singer projection, as in Section 3:*

$$\begin{aligned}\mathrm{Ch}(\mathrm{Ind}_P(\widehat{\eth})) =& \frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{\widehat{M}/B} \widehat{A}(\widehat{M}/B)\,\mathrm{Ch}'(E) \\ & - \frac{1}{2}(\widehat{\eta} + \mathrm{Ch}(\mathrm{null}(\eth^0)) \text{ in } H^*(B).\end{aligned} \tag{16.20}$$

Appendix: Families of pseudodifferential operators

Families of pseudodifferential operators

The algebra of b-pseudodifferential operators on the fibres of a fibration $\phi : M \longrightarrow B$, with fibres which are compact manifolds with boundary, is just the calculus described in [20] with smooth dependence on the base variables. Nevertheless we shall briefly describe a direct definition of this fibre calculus, and subsequently extend it to the corresponding heat calculus, since this provides a proof of the smoothness in the base variables of the constructions. Let $\mathcal{V}_{b,\phi}(M)$ be the Lie algebra of smooth vector fields on the fibres of M, which are tangent to the boundary. Then $\mathrm{Diff}^*_{b,\phi}(M)$, the enveloping algebra, is a local $\mathcal{C}^\infty(M)$-module, and in consequence the space of b-differential operators, $\mathrm{Diff}^*_{b,\phi}(M; E)$, is defined for any smooth bundle E over M.

The b-pseudodifferential operators, which include these b-differential operators, are defined by specifying their Schwartz kernels. Let $\phi^{(2)}$:

$M^{(2)} \longrightarrow B$ be the fibre product of the fibration with itself. Thus

$$\begin{aligned} M^{(2)} &= \left\{ (q,q') \in M^2; \phi(q) = \phi(q') \right\}, \\ \phi^{(2)}(q,q') &= \phi(q) = \phi(q'). \end{aligned} \tag{A.1}$$

Clearly this is a fibration with fibres $M_z^{(2)} = M_z^2$. The total space, $M^{(2)}$, is a manifold with corners of codimension two. Let $B^{(2)} \subset M^{(2)}$ be the 'diagonal corner,' namely

$$B_z^{(2)} = \bigsqcup_{i=1}^{N} \left(\bigcup_{z \in B} {H_{i,z}}^2 \right), \ \partial M_z = H_{1,z} \sqcup H_{2,z} \sqcup \cdots \sqcup H_{N,z}$$

with the $H_{i,z}$ the components of ∂M_z. Thus if ∂M_z is connected then $B_z^{(2)}$ is just the corner of $M_z^{(2)}$. The b-stretched product

$$M^2_{b,\phi} = [M^{(2)}; B^{(2)}], \ \beta : M^2_{b,\phi} \longrightarrow M^{(2)}, \tag{A.2}$$

is obtained by blowing up the N components of $B^{(2)}$. The notation for the blow-up of a submanifold is discussed in [20] and more generally in [19]; the construction just amounts to the introduction of polar coordinates in the normal variables to the submanifold. The extra boundary hypersurfaces of $M^2_{b,\phi}$ produced by this blow-up are called the front faces (one for each component of ∂M_z) and denoted $\mathrm{bf}(M^2_{b,\phi})$.

The fibre diagonal of $M^{(2)}$, just the intersection of the diagonal of M^2 with $M^{(2)}$, lifts to an interior p-submanifold $\Delta_{b,\phi} \subset M^2_{b,\phi}$. By the lift is meant the closure in $M^{(2)}$ of the preimage of the complement of $B^{(2)}$:

$$\Delta_{b,\phi} = \mathrm{cl}\left(\beta^{-1}(\Delta \setminus B^{(2)} \right) \cong M. \tag{A.3}$$

That $\Delta_{b,\phi}$ is an interior p-submanifold means that each point of $\Delta_{b,\phi}$ has a neighbourhood in $M^2_{b,\phi}$ of the form $\Omega_1 \times \Omega_2$ where the point is (p_1, p_2), $\Omega_1 \times \{p_2\}$ is a neighbourhood in $\Delta_{b,\phi}$ and the Ω_i are manifolds with corners.

The calculus of b-pseudodifferential operators, acting on sections of the metric half-density bundle ${}^b\Omega^{\frac{1}{2}}$, consists of three pieces corresponding to various conormal singularities at the lifted diagonal and at the boundary. We shall use the following notation for conormal distributions.

If X is a manifold with corners and Y is an interior p-submanifold, the space of distributions, $I^m(X,Y)$, on X which are conormal to Y and of order m may be defined as the restriction to X of the corresponding space for a manifold without boundary and a closed embedded submanifold, see for example [17, Chapter 18], by doubling across the boundary hypersurfaces.

The closely related spaces, $\mathcal{A}^{\delta-}(X)$, of conormal functions with respect to the boundary of a manifold with corners can be defined by iterative estimates. Recall that $\mathcal{V}_b(X)$ is the Lie algebra of those smooth vector fields on X which are tangent to the boundary, and $\mathrm{Diff}^*_b(X)$ is the algebra of differential operators generated by it. Let ρ_{tb} be a 'total boundary defining function' which is to say the product over the boundary hypersurfaces of X of defining functions for the hypersurfaces. Then

$$\begin{aligned} \mathcal{A}^{\delta}(X) &= \left\{ u \in \rho^{\delta} L^{\infty}(X); \mathrm{Diff}^*_b(X) u \subset \rho^{\delta} L^{\infty}(X) \right\}, \\ \mathcal{A}^{\delta-}(X) &= \bigcup_{\epsilon>0} \mathcal{A}^{\delta-\epsilon}(X). \end{aligned} \tag{A.4}$$

Certainly $\mathcal{A}^{\delta}(X) \subset \mathcal{A}^{\delta-}(X)$, and the effect of weakening the filtration in this way is that the same space results by replacing $L^{\infty}(X)$ by other spaces, such as $L^2_b(X)$.

We also need to consider hybrid spaces between these conormal spaces and $\mathcal{C}^{\infty}(X)$. Let H be a collection of boundary hypersurfaces of X. Then X can always be doubled across the elements of H, successively, and the resulting manifolds are diffeomorphic independently of the order chosen. Let X_H be the result of this doubling. Then

$$\mathcal{A}^{\delta-}_H(X) = \mathcal{A}^{\delta-}(X_H) \restriction X \tag{A.5}$$

is well-defined as a space of functions on the interior of X, and its elements are smooth up to the boundary hypersurfaces in H.

Both spaces $I^m(X,Y)$ and $\mathcal{A}^{\delta-}_H(X)$ are $\mathcal{C}^{\infty}(X)$ modules, so the corresponding sections of any $\mathcal{C}^{\infty}$ vector bundle, E, over X can be defined as a tensor product

$$\begin{aligned} I^m(X,Y;E) =& I^m(X,Y) \otimes_{\mathcal{C}^{\infty}(X)} \mathcal{C}^{\infty}(X;E), \\ \mathcal{A}^{\delta-}_H(X;E) =& \mathcal{A}^{\delta-}_H(X) \otimes_{\mathcal{C}^{\infty}(X)} \mathcal{C}^{\infty}(X;E). \end{aligned} \tag{A.6}$$

The same spaces may be defined simply by localizing the definitions above to open sets on which the bundle is trivial.

The first part of the calculus, the small calculus, is defined in terms of the space of kernels with one-step polyhomogeneous conormal singularity at the lifted diagonal:

$$\begin{aligned} &\Psi^m_{b,\phi}(M;{}^b\Omega^{\frac12}) \\ &= \Big\{K \in I^m_{\rm os}(M^2_{b,\phi},\Delta_{b,\phi};{}^b\Omega^{\frac12}_{\rm fib}); K\equiv 0 \text{ at } \partial\big(M^2_{b,\phi}\setminus \mathrm{bf}\,\big)\Big\}. \end{aligned} \tag{A.7}$$

Here ${}^b\Omega^{\frac12}_{\rm fib}$ is the bundle of fibre b-half-densities on $M^2_{b,\phi}$ and can be naturally identified with

$${}^b\Omega^{\frac12}(M^2_{b,\phi})\otimes(\phi^{(2)})^*(\Omega^{-\frac12}B). \tag{A.8}$$

Since $\Delta_{b,\phi}$ only meets the boundary of $M^2_{b,\phi}$ in the interior of $\mathrm{bf}(M^2_{b,\phi})$, the elements of the conormal space $I^m_{\rm os}(M^2_{b,\phi},\Delta_{b,\phi})$ are smooth up to the other boundary faces, so the vanishing of the Taylor series involved in the definition (A.7) is meaningful. The other two pieces of the full calculus (with bounds, which is all that we consider here) are defined similarly. The residual terms, for a given order δ, are the kernels which are conormal up to the boundary:

$$\Psi^{-\infty,\delta}_{\phi}(M;{}^b\Omega^{\frac12}) = \mathcal{A}^{\delta-}(M^{(2)};{}^b\Omega^{\frac12}_{\rm fib}). \tag{A.9}$$

The boundary terms are intermediate between the small calculus and the residual terms. They have conormal regularity of the given order, δ, up to the boundary hypersurfaces of $M^2_{b,\phi}$ except the front face, up to which they are smooth;

$$\Psi^{-\infty,\delta}_{b,\phi}(M;{}^b\Omega^{\frac12}) = \mathcal{A}^{\delta-}_{\rm bf}(M^2_{b,\phi};{}^b\Omega^{\frac12}_{\rm fib}). \tag{A.10}$$

The blow-down map $\beta: M^2_{b,\phi}\longrightarrow M^{(2)}$ is an isomorphism of the interiors, so the kernels do indeed define maps and we have:

Proposition 18. *For any $\mu\in\mathbb{R}$ each element of $\Psi^*_{b,\phi}(X;E)$ defines a continuous operator on $\mathcal{A}^\mu(X;E)$, and if $\mu+\delta>0$ the elements of $\Psi_{b,\phi}{-\infty},\delta(X;E)$ and $\Psi^{-\infty,\delta}_{\phi}(X;E)$ define operators from $\mathcal{A}^\mu(X;E)$ to $\mathcal{A}^{\min(\mu,\delta)}(X;E)$ and $\mathcal{A}^\delta(X;E)$ respectively.*

The full calculus of fibre b-pseudodifferential operators is the sum:

$$\begin{aligned} \Psi^{m,\delta}_{b,\phi}(M;E) =&\Psi^m_{b,\phi}(M;E) + \Psi^{-\infty,\delta}_{b,\phi}(M,E) \\ &+ \Psi^{-\infty,\delta}_{\phi}(M;E). \end{aligned} \tag{A.11}$$

By Proposition 18 these can be regarded as operators from $\mathcal{A}^{\mu}(X;E)$ to $\mathcal{A}^{\mu'}(X;E)$ provided $\mu+\delta>0$ and $\mu'=\min(\mu,\delta)$. If $\delta>0$ they operate on $\mathcal{A}^{\mu}(X;E)$ for any $\mu\in\mathbb{R}$ with $|\mu|<\delta$, and then these spaces do indeed form algebras. More precisely

Theorem 4. *The spaces $\Psi^{-\infty,\delta}_{b,\phi}(M;E)$ and $\Psi^{-\infty,\delta}_{\phi}(M;E)$ are two-sided modules over the small calculus $\Psi^{*}_{b,\phi}(M;E)$ and, for any $\delta>0$, $\Psi^{-\infty,\delta}_{\phi}(M;E)$ is a two-sided module over $\Psi^{-\infty,\delta}_{b,\phi}(M;E)$. Furthermore $\Psi^{*}_{b,\phi}(M;E)$ is an algebra and for $\delta>0$*

$$\begin{aligned}\Psi^{-\infty,\delta}_{b,\phi}(M;E)\circ\Psi^{-\infty,\delta}_{b,\phi}(M;E)&\\ \subset\Psi^{-\infty,\delta}_{b,\phi}(M;E)+\Psi^{-\infty,\delta}_{\phi}(M;E);&\end{aligned}\tag{A.12}$$

$$\Psi^{-\infty,\delta}_{\phi}(M;E)\circ\Psi^{-\infty,\delta}_{\phi}(M;E)\subset\Psi^{-\infty,\delta}_{\phi}(M;E).$$

For $m\le 0$ and $\delta>0$ all elements of $\Psi^{m,\delta}_{b,\phi}(M;E)$ are bounded on $\rho^{\mu}L^2_b(M;E)$ for any $|\mu|<\delta$, and the elements of $\Psi^{-\infty,\delta}_{\phi}(M;E)$ are of trace class.

Proof. These are just smoothly parametrized version of composition results proved in [20]; alternative, more direct, proofs are given in [19]. We shall only outline the proof here; the same approach will be used below for composition with an element of the heat calculus.

Consider the first part of (A.12). The composite kernel will be written as the push-forward of the product of the kernels of the two factors. To do so consider the triple b-stretched fibre product, $M^3_{b,\phi}$. This is defined from the triple fibre product:

$$M^{(3)}=\left\{(p,p',p'')\in M^3;\phi(p)=\phi(p')=\phi(p'')\right\}$$

with natural projection $\phi:M^{(3)}\longrightarrow B$, by the blow-up of four submanifolds. The first of these is $B^{(3)}$, the union over boundary hypersurfaces $H\subset M$ of the intersection with $M^{(3)}$ of the products H^3. The remaining three are double diagonals of the same type. Thus $B^{(2)}_O\subset M^{(3)}$ is, for $O=F,S$ or C the 'diagonal corner' in two of the three factors of M, respectively the second two factors for $O=F$, the first two for $O=S$ and the outer two factors for $O=C$. The the triple b-stretched fibre product is

$$M^3_{b,\phi}=[M^{(3)};B^{(3)};B^{(2)}_F;B^{(2)}_S;B^{(2)}_C].\tag{A.13}$$

This notation means that the submanifolds are blown up in order from the left. Once $B^{(3)}$ is blown up, the other three submanifolds are disjoint, so the order amongst them is immaterial. Furthermore for any choice of $O = F, S$ or C the order of blow-up of $B^{(3)}$ and $B^{(2)}_O$ can be interchanged so that, for example,

$$M^3_{b,\phi} = [M^{(3)}; B^{(2)}_F; B^{(3)}; B^{(2)}_S; B^{(2)}_C]. \tag{A.14}$$

Since $B^{(2)}_F$ is just the fibre product of M and $B^{(2)}$ in $M^{(3)}$, this shows that there is a natural projection

$$\tilde{\pi}^{(3)}_{b,O} : M^3_{b,\phi} \longrightarrow M^2_{b,\phi} \tag{A.15}$$

for the three choices of O. These three stretched projections give a commutative diagramme:

$$\begin{array}{ccccc} & & M^2_{b,\phi} & & \\ & & \uparrow {\scriptstyle \tilde{\pi}^{(3)}_{b,C}} & & \\ & & M^3_{b,\phi} & & \\ & {\scriptstyle \tilde{\pi}^{(3)}_{b,F}} \swarrow & & \searrow {\scriptstyle \tilde{\pi}^{(3)}_{b,S}} & \\ M^2_{b,\phi} & & & & M^2_{b,\phi} \end{array} \tag{A.16}$$

and over the interior this is just the obvious diagramme for the fibre products $M^{(3)}$ and $M^{(2)}$.

Using these maps the kernel of the composite operator $C = A \circ B$ can be written

$$C = (\tilde{\pi}^{(2)}_{b,C})_* [(\tilde{\pi}^{(2)}_{b,S})^* A \cdot (\tilde{\pi}^{(2)}_{b,F})^* B]. \tag{A.17}$$

All three of the stretched projections in (A.16) are b-fibrations in the sense explained in [18]. Thus the pull-back and push-forward results explained there apply and application of them leads to (A.12). q.e.d.

The construction of the resolvent family of a family of self-adjoint elliptic differential operators of second order can be carried out very much as in [20]. We need the following extension involving the addition of a b-pseudodifferential term of order $-\infty$.

Proposition 19. *If* $P \in \mathrm{Diff}^2_{b,\phi}(M;E)+\Psi^{-\infty}_{b,\phi}(M;E)$ *is elliptic, formally self-adjoint with respect to some positive element of* $\mathcal{C}^\infty(M;{}^b\Omega_{\mathrm{fib}})$ *and inner product on* E *and has positive diagonal principal symbol, then it is self-adjoint and its resolvent extends to a meromorphic family*

$$(P-z)^{-1} \in \Psi^{-2,\delta}_{b,\phi}(M;E) \tag{A.18}$$

in any open set $\Omega \subset \mathbb{C}$ *on which the indicial roots of* $P-z$ *have imaginary parts greater than* δ*. The only poles in such a region are true* L^2 *eigenvalues of finite multiplicity.*

Taking the regular value of the resolvent family at $z=0$ gives:

Corollary 5. *With the assumptions of Proposition 19 and assuming further that the indicial family of* P *has no singular values with imaginary part in the interval* $[-\delta,\delta]$*, the operator* P *has a generalized inverse* $Q \in \Psi^{-2,\delta}_{b,\phi}(M;E)$ *such that*

$$P\circ Q-\mathrm{Id}=Q\circ P-\mathrm{Id}=\Pi_0 \in \Psi^{-\infty,\delta}_{\phi}(M;E) \tag{A.19}$$

is the orthogonal projection onto the null space, in L^2*, of* P*.*

To prove these two results it is convenient (although by no means essential) to use the properties of the heat kernel, which in any case we need to discuss. Thus we shall turn to the fibre heat calculus corresponding to the fibre b-pseudodifferential operators. As in the single fibre case discussed extensively in [20] the space on which the heat kernel takes a simple form for all finite times is obtained from the product $[0,\infty)\times M^2_{b,\phi}$ by the parabolic blow-up of $B_h=\{0\}\times\Delta_{b,\phi}$, with the parabolic direction being the time variable:

$$\begin{aligned} M^2_{\eta,\phi}&=[[0,\infty)\times M^2_{b,\phi};B_h,dt],\\ \beta_h: M^2_{\eta,\phi}&\longrightarrow[0,\infty)\times M^2_{b,\phi}. \end{aligned} \tag{A.20}$$

Such parabolic blow-up corresponds to the introduction of 'parabolic polar coordinates' in the normal variables to the submanifold blown up. In this case H_h is locally fixed by $t=0$, $z=z'$ and appropriate polar coordinates are:

$$r=(|z-z'|^4+t^2)^{\frac12},\ \tau=\frac{t}{r^2},\ \omega=\frac{(z-z')}{r}.$$

The new boundary hypersurface produced by this blow-up will be denoted tf and that arising from the blow-up defining $M^2_{b,\phi}$ again as bf .

The main part of the fibre heat calculus can then be defined as in [20]. Thus, for operators on sections of ${}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}}$ we consider the kernels

$$
\begin{aligned}
&\Psi^p_{\eta,\phi}(M;{}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}}) \\
&\quad = \left\{ K \in \rho_{\mathrm{tf}}^{-\frac{1}{2}(n+3)-p}\mathcal{C}^\infty_{\mathrm{ev}}(M^2_{\eta,\phi};{}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}}); K \equiv 0 \text{ at } \partial M^2_{b,\phi} \setminus (\mathrm{tf}\cup\mathrm{bf}) \right\}, \\
&\qquad\qquad p \in -\mathbb{N},
\end{aligned} \tag{A.21}
$$

with n the dimension of the fibres of $\phi : M \longrightarrow B$. We refer to [20] for a discussion of the notion of odd and even Taylor series at tf. For p even this evenness condition can be replaced by the requirement that the operators defined by these kernels are maps

$$
\begin{aligned}
\Psi^p_{\eta,\phi}(M;{}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}}) \ni A : \mathcal{C}^\infty(M;{}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}}) \\
\longrightarrow \mathcal{C}^\infty([0,\infty)\times M;{}^b\Omega_{\mathrm{fib}}^{\frac{1}{2}})
\end{aligned} \tag{A.22}
$$

rather than the image being in the space of smooth functions of $t^{\frac{1}{2}}$. As before the extension to a general bundle E can be accomplished either by use of a partition of unity to localize to open sets over which the bundles are trivial or else by taking tensor products, as in (A.6).

In addition to these main terms we need to consider 'residual terms' which arise from the pseudodifferential perturbations (of order $-\infty$). By this we mean the space

$$
t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)) \text{ for } k \in \mathbb{N}_0 = \{0,1,\dots\}.
$$

These act as t-convolution operators, together with the fibre action of the b-pseudodifferential operators. The combination of the two terms, for any $k \geq 0$ and $p < 0$:

$$
\begin{aligned}
\Psi^{p,k}_{\eta,\phi}(M;E) = &\Psi^p_{\eta,\phi}(M;E) \\
&+ t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E))
\end{aligned} \tag{A.23}
$$

forms an algebra. For the discussion of the heat kernel of the perturbed superconnection we do not need to compose elements in the small calculus, given by (A.21), so we shall simply show that the second part of (A.23) forms an ideal over the first part.

Proposition 20. *For any fibration with fibres which are compact manifolds with boundary, any $p \in -\mathbb{N}$ and any $k \in \mathbb{N}_0$*

(A.24)

$$\begin{aligned}
&\Psi^{2p}_{\eta,\phi}(M;E) \circ t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)) \subset t^{k-p}\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)),\\
&t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)) \circ \Psi^{2p}_{\eta,\phi}(M;E) \subset t^{k-p}\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)),\\
&\Psi^{-\infty}_{b,\phi}(M;E) \circ \Psi^{2p}_{\eta,\phi}(M;E) \subset t^{-p-1}\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E)).
\end{aligned}$$

Proof. It suffices to prove the last statement, in which the element of $\Psi^{-\infty}_{b,\phi}(M;E)$ acts with t as a parameter. This is equivalent to the convolution of $\delta(t)$ so behaves as an element of $t^{-1}\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E))$. More precisely writing the convolution action of

$$A \in t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(M;E))$$

as

$$Au(t,\cdot) = \int_0^t A(t-s)u(s,\cdot)ds$$

allows the composite $C = A \circ B$, with $B \in \Psi^{2p}_{\eta,\phi}(M;E)$ to be written as

$$C(t) = \int_0^t \left(s^k A'(s) \circ B\right)(t-s)ds, \tag{A.25}$$

where $A' \in t^k\mathcal{C}^\infty([0,\infty);\Psi^{-\infty}_{b,\phi}(X;E))$. Given the continuity of the last composition formula in (A.24) we obtain the second formula. The first then follows from the second by taking adjoints.

Thus we turn to the proof of the third formula. As with the composition formula for the small b-pseudodifferential calculus in [19], discussed briefly above, we formulate the composition law geometrically in the following diagramme of spaces and maps:

$$\begin{array}{ccc} & [0,\infty)\times M^2_{b,\phi} & \\ & \uparrow \tilde\pi^{(3)}_{S\eta,C} & \\ & M^3_{S,\eta,\phi} & \\ \swarrow \tilde\pi^{(3)}_{S\eta,F} & & \tilde\pi^{(3)}_{S\eta,S} \searrow \\ M^2_{b,\phi} & & M^2_{\eta,\phi} \end{array} \tag{A.26}$$

The maps are all lifts of projections, with the central 'triple' space being defined by parabolic blow-up:

$$\begin{aligned} &M^3_{S,\eta,\phi} = [[0,\infty)\times M^3_{b,\phi}; B_S, dt], \\ &\beta^{(3)}_h : M^3_{S,\eta,\phi} \longrightarrow [0,\infty)\times M^3_{b,\phi}. \end{aligned} \tag{A.27}$$

Here t is the variable in the first factor, and B_S is the product of $\{0\}$ and the lift to $M^3_{b,\phi}$ of the diagonal in the second two factors of M. The map $\tilde\pi^{(3)}_{S\eta,F}$ is then defined as the composite of the parabolic blow-down map, projection off the t-factor followed by the stretched projection from $M^3_{b,\phi}$ to the space $M^2_{b,\phi}$ corresponding to the left two factor of M :

$$\tilde\pi^{(3)}_{S\eta,F} = \tilde\pi^{(3)}_{b,F}\circ\pi_t\circ\beta^3_h. \tag{A.28}$$

The map $\tilde\pi^{(3)}_{S\eta,C}$ is defined similarly, except without projection off the t factor and with the stretched projection corresponding to the outer two factors of M :

$$\tilde\pi^{(3)}_{S\eta,C} = \tilde\pi^{(3)}_{b,C}\circ\beta^3_h. \tag{A.29}$$

The definition of the remaining map is a little more subtle, since it involves the commutation of blow-ups. The lifted partial diagonal blown up in (A.27) is disjoint from two of the four (sets of) hypersurfaces of $M^3_{b,\phi}$ produced by blow-up, namely the two others arising from the blow-ups of the other two partial diagonals. These two hypersurfaces can therefore be blown down. While the triple front face of $M^3_{b,\phi}$ is not disjoint from B_S, it does meet B_S transversally. Thus the blow-up orders can be exchanged, which is to say that there is a natural map $\tilde{\pi}^{(3)}_{S\eta,S}$ as in (A.26), given as a product of blow-down maps and a projection. This completes the definition of the diagramme (A.26).

Next we note that the three maps in (A.26) are all b-fibrations, although we only really need this for $\tilde{\pi}^{(3)}_{S\eta,C}$. This can be seen directly using the transversality theorems of [19], or the algebraic condition on the b-differential can be checked directly.

From these facts the composition formula follows directly. The kernel of the composite operator can be written

$$C = (\tilde{\pi}^{(3)}_{S\eta,C})_*[(\tilde{\pi}^{(3)}_{S\eta,S})^* A \cdot (\tilde{\pi}^{(3)}_{S\eta,F})^* B]. \tag{A.30}$$

The two lower maps in (A.26) are b-maps so the product in (A.30) is certainly polyhomogeneous conormal at all boundary hypersurfaces. It vanishes rapidly at any boundary hypersurface which is mapped into a boundary hypersurface at which the corresponding factor kernel vanishes rapidly, this gives rapid vanishing at all boundary hypersurfaces of $M^3_{S,\eta,\phi}$ except that produced by the blow-up in (A.27) and that arising from the 'triple front face' of $M^3_{b,\phi}$.

To compute the precise order of the resulting kernel we need to calculate Jacobian factors which arise on lifting the densities from the factors. The kernel A is a smooth section of the ${}^b\Omega^{\frac{1}{2}}_{\text{fib}}$ over $M^2_{b,\phi}$. The kernel B is of the form $\rho^b_{\text{tf}} f$ where f is a smooth section of ${}^b\Omega^{\frac{1}{2}}_{\text{fib}}$ for the fibration $M^2_{\eta,\phi} \longrightarrow B$ and $b = -\frac{n}{2} - 1 - 2p$. Recalling that $M^2_{\eta,\phi}$ is defined by the blow-up (A.20) and observing that the b-fibre density bundle for the fibration $[0,\infty) \times M^2_{b,\phi} \longrightarrow B$ lifts naturally under the blow-up as:

$$\beta_h^{*b}\Omega_{\text{fib}} = \rho^n_{\text{tf}}\,{}^b\Omega_{\text{fib}},$$

we can write

$$B \in \rho_{\text{tf}}^{-n-1-2p} \beta_h^{*b}\Omega^{\frac{1}{2}}_{\text{fib}}. \tag{A.31}$$

Under the blow-up of a boundary face, as in the construction of $M^3_{b,\phi}$, the lift of the b-density bundle is naturally isomorphic to the b-density bundle of the lift; the same is true for the fibre density bundles. This means that if we take a smooth positive element $\nu \in \mathcal{C}^\infty(M^2_{b,\phi}; {}^b\Omega^{\frac{1}{2}}_{\text{fib}})$ and multiply it by $|dt|^{\frac{1}{2}}$ this can be regarded as a smooth section of $t^{\frac{1}{2}}\,{}^b\Omega^{\frac{1}{2}}_{\text{fib}}$ for the fibre half-densities bundle for the fibration $[0,\infty) \times M^2_{b,\phi} \longrightarrow B$.

This means that the product is naturally a section

$$(\tilde{\pi}^{(3)}_{S\eta,S})^* A \cdot (\tilde{\pi}^{(3)}_{S\eta,F})^* B \cdot (\tilde{\pi}^{(3)}_{S\eta,C})^* (\nu \cdot |dt|^{\frac{1}{2}}) \\ \in \rho_{\mathrm{tf}}^{-n-2p} (\beta^3_h)^{*b}\Omega_{\mathrm{fib}} = \rho_{\mathrm{tf}}^{-2p\,b}\Omega_{\mathrm{fib}}. \tag{A.32}$$

From the push-forward theorems of [18] (and taking into account the rapid vanishing at most faces which prevents the appearance of logarithmic terms) it follows that this product pushes forward under $\tilde{\pi}^{(3)}_{S\eta,C}$ to a density of the form $t^{-p} g\mathcal{C}^\infty([0,\infty)\times M^2_{b,\phi}; {}^b\Omega_{\mathrm{fib}})$, where g is in principle only smooth as a function of $t^{\frac{1}{2}}$. Taking into account the section $\nu|dt|^{\frac{1}{2}}$ this is just the product of the kernel C and $|dt|^{\frac{1}{2}}$; so the resulting kernel is an element of $\Psi^{-\infty}_{b,\phi}(X; {}^b\Omega^{\frac{1}{2}}_{\mathrm{fib}})$ depending smoothly on $t^{\frac{1}{2}} \in [0,\infty)$ and vanishing to order $-p-1$ at $t=0$. That the composite is actually a smooth function of t is a consequence of the assumption that the Taylor series in (A.21) is even. It follows from the fact that the composite operator must preserve regularity in t since the factors do so. q.e.d.

The 'small' heat calculus in (A.21) is defined so that if

$$P \in \mathrm{Diff}^2_{b,\phi}(M;E)$$

is elliptic, formally self-adjoint and has positive diagonal symbol then

$$\exp(-tP) \in \Psi^{-2}_{\eta,\phi}(M;E). \tag{A.33}$$

For the single fibre case this is shown in [20] and the proof there gives (A.33) with only notational changes. In Section 11 it is shown, using Proposition 20, that the addition of a self-adjoint b-pseudodifferential term of order $-\infty$ just means that heat calculus in (A.33) must be replaced by the larger spaces from (A.23) and then

$$\exp(-tP) \in \Psi^{-2,1}_{\eta,\phi}(M;E). \tag{A.34}$$

Using this we briefly describe the proof of Proposition 19, following the corresponding discussion from [20] closely.

Cutting the heat kernel in (A.34) off smoothly near $t=1$ and taking the Fourier-Laplace transform in t gives an entire parametrix for the resolvent of P :

$$G(z) \in \Psi^{-2}_{b,\phi}(M;E) \text{ with } (P-z)G(z) = \mathrm{Id} - R(z),$$

where $\|R(z)\| \leq C/|z|$ as $|z| \to \infty$ in any sector away from $[0, \infty)$. To deduce from this that the resolvent is in the full b-calculus it is only necessary to solve the indicial problem to remove the indicial operator of the error, using the assumption that there are no real indicial roots, and then to use the bi-ideal property of the residual part of the calculus and analytic Fredholm theory.

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